THE ASYMPTOTIC DIMENSION OF THE GRAND ARC GRAPH IS INFINITE

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Abstract. Let Σ be a compact, orientable surface, and let Γ be a relation on $\pi_0(\partial\Sigma)$ such that the prescribed arc graph $\mathcal{A}(\Sigma,\Gamma)$ is Gromov-hyperbolic and non-trivial. We show that $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) \geq -\chi(\Sigma) - 1$, from which we prove that the asymptotic dimension of the grand arc graph is infinite. More generally, we prove that any connected, Gromov-hyperbolic multiarc and curve graph $\mathcal{M}$ preserved by $\mathrm{PMod}(\Sigma)$ with bounded geometric intersection over edges has $\operatorname{asdim}\mathcal{M} \geq g - \lceil\frac{1}{2}\chi(\Sigma)\rceil$, and that a broad class of multiarc and curve graphs on infinite-type surfaces has infinite asymptotic dimension.

1. Introduction

Let Σ be a compact, orientable surface with boundary, and let Γ be a relation on $\pi_0(\partial\Sigma)$. A simple, essential arc a in Σ is Γ-*allowed* if it joins boundary components in Γ. The Γ-*prescribed arc graph* $\mathcal{A}(\Sigma,\Gamma)$ is the full subgraph of $\mathcal{A}(\Sigma)$ spanned by isotopy classes of Γ-allowed arcs. We assume throughout that $\mathcal{A}(\Sigma,\Gamma)$ is *non-trivial*, *i.e.* $\chi(\Sigma) \leq -1$, $\Sigma \neq \Sigma_0^3$, and $\Gamma \neq \varnothing$.

We suppose $\mathcal{A}(\Sigma,\Gamma)$ is δ-hyperbolic. If $\Sigma = \Sigma_0^4$, then $\mathcal{A}(\Sigma,\Gamma) \subset \mathcal{A}(\Sigma_0^4)$ is a quasi-tree and $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) = 1$. Otherwise, we prove a lower bound:

Theorem 1.1. *If $\mathcal{A}(\Sigma,\Gamma)$ is δ-hyperbolic, then* $-\chi(\Sigma) - 1 \leq \operatorname{asdim}\mathcal{A}(\Sigma,\Gamma)$.

For Ω an infinite-type surface with finite grand splitting, let $\mathcal{G}(\Omega)$ denote the *grand arc graph* on Ω [BNV22]. By applying Theorem 1.1, we obtain the following:

Theorem 1.2. *If $\mathcal{G}(\Omega)$ is non-empty and connected, then* $\operatorname{asdim}\mathcal{G}(\Omega) = \infty$.

More generally, let $\mathcal{M}$ be any connected multiarc and curve graph on a surface Ω that is preserved by $\mathrm{PMod}(\Omega)$, admits a (compact) witness, and in each witness has uniformly bounded geometric intersection over edges.

Theorem 1.3. *If Ω is compact and $\mathcal{M}$ is δ-hyperbolic, then* $\operatorname{asdim}\mathcal{M} \geq g(\Omega) - \lceil\frac{1}{2}\chi(\Omega)\rceil$. *If Ω is infinite-type, then* $\operatorname{asdim}\mathcal{M} = \infty$.

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In Section 2, we use the theory of alignment-preserving maps [DT17] to show that the Gromov boundary $\partial\mathcal{A}(\Sigma,\Gamma)$ contains $\partial\mathcal{A}(\Sigma)$. From results of Gabai [Gab14] and Schleimer [Po17] we obtain a compact subspace $Z \subset \partial\mathcal{A}(\Sigma,\Gamma)$ of dimension $-\chi(\Sigma)-2$. We then prove that $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) \geq \dim Z + 1$, extending a result for proper δ-hyperbolic spaces, whence Theorem 1.1 follows.

In Section 3, we show that witness subsurfaces $W \subset \Omega$ for $\mathcal{G}(\Omega)$ of arbitrarily large complexity admit prescribing relations Γ such that $\mathcal{A}(W,\Gamma)$ quasi-isometrically embeds into $\mathcal{G}(\Omega)$, where Ω is an infinite-type surface with finite grand splitting. In fact, W may be chosen so that either $\mathcal{A}(W,\Gamma)$ has large coarse rank or it is δ-hyperbolic: Theorem 1.2 thus follows from Theorem 1.1 and the monotonicity of asymptotic dimension.

Section 4 generalizes the techniques in Sections 2 and 3 to a broad class of simplicial graphs, called *admissible combinatorial models*, which include prescribed arc graphs, the grand arc graph, the 1-skeleton of the marking complex, and many other multiarc and curve graphs. In addition to tools developed in Section 2, we utilize properties of the hierarchically hyperbolic structure of such graphs in the finite-type setting [Kop23a]. Theorem 1.3 follows from the analogous statements for admissible combinatorial models.

Remark. For the reader interested in only Theorem 1.3 (which does imply Theorem 1.2 and a weaker version of Theorem 1.1, albeit with more technology than necessary), it suffices to read Sections 2.1 and 4.

1.1. **Background.** An orientable surface Ω has *infinite topological type* if its fundamental group is not finitely generated, or equivalently if $\operatorname{int}(\Omega)$ has infinite genus or infinitely many punctures (we typically assume $\partial\Omega = \varnothing$). Beginning with a 2009 blog post of Calegari [Cal09], mapping class groups of infinite-type surfaces have been objects of considerable contemporary study: see [AV20, CPV21] for surveys of recent results and open problems.

An infinite-type surface Ω is classified by its genus and *end space*, which is obtained as the inverse limit of the complementary components of a compact exhaustion [Ric63]; its mapping class group $\operatorname{Mod}(\Omega)$ is a non-compactly generated Polish group. Given mild assumptions, Mann and Rafi [MR20] classify when $\operatorname{Mod}(\Omega)$ admits a generating set that is *coarsely bounded (CB)*, or bounded in any left-invariant metric, and hence a well defined quasi-isometry type in the sense of [Ros14]. Mann–Rafi also define a preorder on the ends of Ω corresponding to topological complexity. We denote by $\mathscr{M}(\Omega)$ the non-empty subspace of maximal ends with respect to this preorder.

When $\operatorname{Mod}(\Omega)$ is locally CB (and in particular when it is CB-generated), Bar-Natan and Verberne define the *grand splitting* $\mathcal{S}(\Omega)$, a canonical and $\operatorname{Mod}(\Omega)$-invariant partition of $\mathscr{M}(\Omega)$ into finitely many disjoint sets $E_i \in$

$\mathcal{S}(\Omega)$, each of which is either a singleton or Cantor set. A *grand arc* in Ω is a bi-infinite simple arc converging to ends in distinct sets in the grand splitting [BNV22].

Definition 1.4 (Bar-Natan–Verberne)**.** Let Ω be an infinite-type surface. The *grand arc graph* $\mathcal{G}(\Omega)$ is the simplicial graph with vertices corresponding to isotopy classes of grand arcs and edges determined by disjointness.

The grand arc graph $\mathcal{G}(\Omega)$ is a combinatorial model for Ω which generalizes the ray graph defined by Calegari [Cal09] on $S^2 \setminus$ Cantor set and for surfaces with stable endspace extends the omnipresent arc graph defined by Fanoni–Ghaswala–McLeay [FGM21]. $\mathrm{Mod}(\Omega)$ acts naturally on $\mathcal{G}(\Omega)$ by isometries. Bar-Natan–Verberne classify the δ-hyperbolicity of $\mathcal{G}(\Omega)$ and show that when $\mathcal{G}(\Omega)$ is δ-hyperbolic, the action of $\mathrm{Mod}(\Sigma)$ is quasi-continuous, extends continuously to $\partial\,\mathcal{G}(\Omega)$, and has loxodromic elements.

Notation. We typically denote by Σ a compact, orientable surface, and by Ω an arbitrary orientable surface that may have either finite or infinite topological type.

1.1.1. *Prescribed arc graphs, witnesses.* Prescribed arc graphs were defined by the author in [Kop23b] as combinatorial models of finite-type surfaces that quasi-isometrically embed into $\mathcal{G}(\Omega)$. Excepting trivial cases they are connected and infinite-diameter and their δ-hyperbolicity is fully determined by the prescribing relation Γ:

Theorem 1.5 ([Kop23b, Thm. 1.3])**.** *Assume that $\mathcal{A}(\Sigma,\Gamma)$ is non-trivial. Then if $\chi(\Sigma) \geq -2$ or $\Sigma = \Sigma_0^{n+1}$ and Γ is a n-pointed star then $\mathcal{A}(\Sigma,\Gamma)$ is δ-hyperbolic. Otherwise, $\mathcal{A}(\Sigma,\Gamma)$ is (uniformly) δ-hyperbolic if and only if Γ is not bipartite.*

We note that if $\Gamma \subset \Gamma'$ then every Γ-allowed arc is Γ'-allowed, which induces a simplicial map $\iota : \mathcal{A}(\Sigma,\Gamma) \to \mathcal{A}(\Sigma,\Gamma')$. This map is 3-coarsely surjective [Kop23b, Lem. 2.10]. In particular, since the prescribed arc graph with the complete relation is exactly $\mathcal{A}(\Sigma)$, $\mathcal{A}(\Sigma,\Gamma)$ always coarsely surjects onto $\mathcal{A}(\Sigma)$.

A compact, essential (π_1-injective, non-peripheral) non-pants subsurface is a *witness* for a given combinatorial model if it intersects every vertex. We call a witness $W \subset \Sigma$ for $\mathcal{A}(\Sigma,\Gamma)$ a Γ-*witness.*

1.1.2. *Boundaries of non-proper δ-hyperbolic spaces.* In general, if $\mathcal{A}(\Sigma,\Gamma)$ is non-trivial then it is non-proper, and likewise for any admissible combinatorial model with sufficient complexity. For a geodesic δ-hyperbolic space X, by ∂X we always mean the sequential boundary of X; when X is non-proper, ∂X may be non-compact. In this setting, ∂X does not coincide with the geodesic boundary, but is instead homeomorphic to the quasi-geodesic boundary [Has22]. We will make use of the following statement by Hasegawa, from a construction of Kapovich–Benakli [KB02, Rmk. 2.16]:

Remark 1.6 ([Has22, Prop. 4]). Fixing $x_0 \in X$, for any $z \in \partial X$ there exists a $(1+4\delta, 12\delta)$-quasi-geodesic ray $\rho : [0, \infty) \to X$ based at x_0 with $[\rho(n)] = z$.

Any quasi-isometry between geodesic δ-hyperbolic spaces $X \to Y$ extends to a map $X \cup \partial X \to Y \cup \partial Y$ that restricts to a homeomorphism on the boundaries (*e.g.* applying the proof of [DK18, Thm. 11.108]). Given $x, y \in X \cup \partial X$, let $(x|y)_{x_0}$ denote their Gromov product at x_0. We occassionally omit the basepoint, which is changeable up to bounded error.

1.1.3. *Ending laminations.* Let $\chi(\Sigma) \leq -1$, hence fix a (finite-area) hyperbolic metric for Σ with geodesic boundary. We recall that a *geodesic lamination* on Σ is a closed subset $L \subset \Sigma$ which decomposes (in fact, uniquely) into pair-wise disjoint simple geodesic leaves. L is *minimal* if it has no proper sublaminations, or equivalently, if every leaf is dense in L.

Definition 1.7. Given a connected subspace $X \subset \Sigma$ with non-trivial π_1-image, if $Y \subset \Sigma$ is the smallest essential subsurface containing X up to isotopy, then Y is *filled* by X. If $Y = \Sigma$, then X is *filling.*

Definition 1.8. The space of *ending laminations* $\mathcal{EL}(\Sigma)$ is the set of filling minimal laminations on Σ, equipped with the coarse Hausdorff topology. Similarly, let $\mathcal{EL}_0(\Sigma)$ denote the space of minimal laminations that fill a subsurface containing $\partial\Sigma$, again with the coarse Hausdorff topology.

$\mathcal{EL}(\Sigma)$ and $\mathcal{EL}_0(\Sigma)$ give explicit descriptions for the hyperbolic boundaries of $\mathcal{C}(\Sigma)$ and $\mathcal{A}(\Sigma)$, respectively (see [Kla18] and [Po17]):

Theorem 1.9 (Klarreich, Schleimer)**.** *$\mathcal{EL}(\Sigma) \cong \partial\mathcal{C}(\Sigma)$ and $\mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma)$.*

1.1.4. *Markings.* In Section 4, we will make use of *markings* on surfaces in the sense of [MM00]. For an essential simple closed curve $a \subset \Omega$, let $\mathcal{C}(a)$ denote the curve graph of the annulus with core a and π_a the corresponding (set-valued) subsurface projection.

Definition 1.10. A *marking* $\mu = \{(a_i, t_i)\}$ on a surface Ω is an essential simple multicurve $\{a_i\}$, denoted $\operatorname{base}\mu$, along with a collection of (possibly empty) diameter 1 subsets $t_i \subset \mathcal{C}(a_i)$; for $a_i \in \operatorname{base}\mu$, let $\operatorname{trans}_\mu(a_i) = t_i$ denote the associated *transversal.*

A marking μ is *complete* if $\operatorname{base}\mu$ is a pants decomposition and every transversal is non-empty. If μ is complete and for each component $(a, t) \in \mu$ $t = \pi_a b$ for some simple closed curve $b \neq a$ disjoint from $\operatorname{base}\mu \setminus \{a\}$ that intersects a minimally, then μ is *clean.*

Let $\Delta \subset \Omega$ be an essential, non-pants subsurface. Like multicurves, markings have a subsurface projection $\pi_\Delta(\mu) \subset \mathcal{C}(\Delta)$. If Δ is an annulus parallel to some $a \in \operatorname{base}\mu$, then $\pi_\Delta(\mu) := \operatorname{trans}_\mu(a) \subset \mathcal{C}(\Delta)$. Otherwise, $\pi_\Delta(\mu) := \pi_\Delta(\operatorname{base}\mu)$. We say Δ intersects μ if and only if $\pi_\Delta(\mu) \neq \varnothing$. For an essential simple closed curve $c \subset \Omega$, again let $\pi_c(\mu)$ denote the projection to the annulus with core c.

Definition 1.11. Let μ, ν be two markings on Ω. Then their *geometric intersection number* $i(\mu, \nu)$ is defined as follows:

$$i(\mu,\nu) := i(\operatorname{base}\mu, \operatorname{base}\nu) + \sum_{a \in \operatorname{base}\mu \cup \operatorname{base}\nu} \operatorname{diam}_{\mathcal{C}(a)}(\pi_a\mu \cup \pi_a\nu)$$

1.1.5. *Alignment-preserving maps.* We briefly recall the theory of alignment-preserving maps from [DT17]. Let X be a geodesic metric space. Then a triple $(x, y, z) \in X^3$ is K*-aligned* if $d(x, y) + d(y, z) \leq d(x, z) + K$. A Lipschitz map between geodesic metric spaces $f : X \to Y$ is *coarsely alignment preserving* if there exists $K \geq 0$ for which f maps any 0-aligned triple in X to a K-aligned triple in Y.

Suppose that $f : X \to Y$ is a coarsely alignment preserving map between geodesic δ-hyperbolic spaces. Then we define $\partial_Y X \subset \partial X$ to be

$$\partial_Y X := \{[\gamma] \in \partial X \,|\, \gamma : \mathbb{R}^+ \to X \text{ quasi-geodesic, } \operatorname{diam}_Y(f\gamma(\mathbb{R}^+)) = \infty\}.$$

Theorem 1.12 (Dowdall–Taylor, [DT17, Thm. 3.2])**.** *Let $f : X \to Y$ be a coarsely surjective, coarsely alignment preserving map between geodesic δ-hyperbolic spaces. Then f admits an extension to a homeomorphism $\partial f : \partial_Y X \to \partial Y$ such that if $x_n \to \omega \in \partial_Y X$, then $f(x_n) \to \partial f(\omega)$.*

2. Asymptotic dimension of $\mathcal{A}(\Sigma, \Gamma)$

When $\Sigma = \Sigma_0^4$, then $\mathcal{A}(\Sigma,\Gamma) \subset \mathcal{A}(\Sigma_0^4)$ is an infinite-diameter connected subgraph of a quasi-tree, hence likewise a quasi-tree: $\operatorname{asdim}\mathcal{A}(\Sigma,\Gamma) = 1$. For $\Sigma \neq \Sigma_0^4$, we first prove $\mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma) \subset \partial\mathcal{A}(\Sigma,\Gamma)$ for Γ not bipartite.

Lemma 2.1. *If Γ is not bipartite then for any $\Gamma' \supset \Gamma$ the induced coarse surjection $\iota : \mathcal{A}(\Sigma,\Gamma) \to \mathcal{A}(\Sigma,\Gamma')$ is uniformly coarsely alignment-preserving.*

Proof. We first claim that if Γ is not bipartite, then geodesics in $\mathcal{A}(\Sigma,\Gamma)$ are uniformly (independent of Γ) Hausdorff close to unicorn paths with coarsely the same endpoints, and *vice versa.* If Γ is not bipartite and if $\Sigma = \Sigma_1^2$ then Γ is not two loops, then the claim holds by [Kop23b, §3]. If instead $\Sigma = \Sigma_1^2$ and $\Gamma = \ell_1 \cup \ell_2$ is two loops, then $\iota : \mathcal{A}(\Sigma,\ell_1) \to \mathcal{A}(\Sigma,\Gamma)$ is a quasi-isometry by [Kop23b, Lem. 5.2] and we apply the Morse lemma. We observe that if Γ is not bipartite then neither is Γ'.

We claim that for any geodesic γ between $a, b \in \mathcal{A}(\Sigma, \Gamma)$, $\iota\gamma$ is uniformly Hausdorff close to a geodesic between $\iota(a), \iota(b)$, whence the proof follows. Let γ' be a unicorn path close to γ, in the sense above. ι is Lipschitz, hence $\iota\gamma, \iota\gamma'$ are close; since $\iota\gamma'$ is a unicorn path in $\mathcal{A}(\Sigma, \Gamma')$, choose a geodesic γ'' close to $\iota\gamma'$. By the Morse lemma, there exists a geodesic γ''' between $\iota(a), \iota(b)$ that is close to γ'', hence close to $\iota\gamma$. □

Applying Theorem 1.12, we obtain the desired embedding.

Corollary 2.2. *If Γ is not bipartite and $\Gamma' \supset \Gamma$, then there exists an embedding $(\partial\iota)^{-1} : \partial\mathcal{A}(\Sigma, \Gamma') \to \partial\mathcal{A}(\Sigma, \Gamma)$.* □

By [Kop23b, §5], if $\Sigma \neq \Sigma_0^4$ and $\mathcal{A}(\Gamma, \Sigma)$ is δ-hyperbolic, then (i) $\Sigma = \Sigma_0^{n+1}$ and Γ is an n-pointed star, (ii) $\Sigma = \Sigma_1^2$ and Γ is a non-loop edge, or (iii) Γ is not bipartite. In case (i), by [Kop23b, Lem. 5.4] $\mathcal{A}(\Sigma, \Gamma)$ is quasi-isometric to $\mathcal{A}(\Sigma, \ell_0)$, where ℓ_0 is a single loop and hence not bipartite. Thus for cases (i) and (iii), Corollary 2.2 implies $\partial\mathcal{A}(\Sigma) \subset \partial\mathcal{A}(\Sigma, \Gamma)$. In case (ii), every Γ-witness is in fact a witness for the usual arc graph: by [Kop23a] $\mathcal{A}(\Sigma, \Gamma)$ and $\mathcal{A}(\Sigma)$ have the same quasi-isometry type, hence $\partial\mathcal{A}(\Sigma, \Gamma) \cong \partial\mathcal{A}(\Sigma)$.

Proposition 2.3. *Let $\Sigma \neq \Sigma_0^4$ and $\mathcal{A}(\Sigma, \Gamma)$ be δ-hyperbolic. $\partial\mathcal{A}(\Sigma) \cong \mathcal{EL}_0(\Sigma)$ embeds canonically into $\partial\mathcal{A}(\Sigma, \Gamma)$.* □

2.1. **A lower bound.** From [Gab14], we have the following:

Theorem 2.4 (Gabai)**.** *Let S be the $(n + 4)$-times punctured sphere for $n \geq 0$. Then $\mathcal{EL}(S)$ is homeomorphic to the n-dimensional Nöbeling space.*

For any Σ with $\chi(\Sigma) \leq -2$, let $n = n(\Sigma) = -\chi(\Sigma) - 2$ and let Γ be a prescribing relation such that $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic. We may choose an essential $(n + 4)$-punctured sphere S that contains all of the punctures of Σ, thus $\mathcal{EL}(S) \subset \mathcal{EL}_0(\Sigma) \cong \partial\mathcal{A}(\Sigma)$. Then applying Proposition 2.3 and Theorem 2.4, $\partial\mathcal{A}(\Sigma, \Gamma)$ contains the n-dimension Nöbeling space, and in particular, a compact subspace $Z \subset \mathcal{EL}(S)$ of topological dimension n by the universal embedding property of Nöbeling spaces [Nöb30].

For the remainder of the section, we will prove the following generalization of a result for proper δ-hyperbolic spaces (*e.g.* [BL08, Prop. 6.2]):

Proposition 2.5. *Let X be a geodesic δ-hyperbolic space with $Z \subset \partial X$ compact. Then* $\operatorname{asdim} X \geq \dim Z + 1$.

Since $\partial\mathcal{A}(\Sigma, \Gamma)$ contains a $n(\Sigma)$-dimensional compact subspace for $\chi(\Sigma) \leq -2$, Theorem 1.1 follows (vacuously for $\chi(\Sigma) > -2$).

By δ-hyperbolic, we mean that geodesic triangles are δ-slim. Let X be a geodesic δ-hyperbolic space and let $Z \subset \partial X$ be compact. A metric d :

$\partial X \times \partial X \to [0,\infty)$ is *visual* if there exist k_1, k_2 and $a > 0$ such that

$$k_1 a^{-(\xi|\xi')} \leq d(\xi,\xi') \leq k_2 a^{-(\xi|\xi')}.$$

Such metrics always exist [BH99, Prop. III.H.3.21] and are compatible with the usual topology on the (sequential) boundary: $d(\xi_i,\xi) \to 0$ if and only if $(\xi_i|\xi) \to \infty$, which is equivalent to $\xi_i \to \xi$.

Notation. Where unambiguous, we denote by $|xx'|$ the distance between $x, x' \in X$ a metric space. Given a specified basepoint $o \in X$, let $|x| := |ox|$.

Definition 2.6. For (Z, d) a bounded metric space, the *hyperbolic cone* $\operatorname{Co} Z$ is the topological cone $Z \times [0,\infty)/Z \times \{0\}$ endowed with the following metric. Let $\mu = \pi/\operatorname{diam}(Z)$. For any $x = (z,t), x' = (z',t') \in \operatorname{Co} Z$, consider a geodesic triangle $\bar{o}\bar{x}\bar{x}' \subset H^2$ with $|\bar{o}\bar{x}| = t, |\bar{o}\bar{x}'| = t'$, and $\angle_{\bar{o}}(\bar{x},\bar{x}') = \mu|zz'|$. Then let $|xx'| := |\bar{x}\bar{x}'|$.

This metric is compatible with the usual topology on $\operatorname{Co} Z$. In addition, $\operatorname{Co} Z$ is δ-hyperbolic, $Z \hookrightarrow \partial \operatorname{Co} Z$ via the geodesic rays $\gamma_z : t \mapsto (z,t)$, and d is visual for $Z \subset \partial \operatorname{Co} Z$ with respect to $\operatorname{Co} Z$ [Buy06, Prop. 6.1]. We fix $o = Z \times \{0\}$ as a basepoint for $\operatorname{Co} Z$. Analogously to [Buy06, Prop. 6.2], we have the following:

Lemma 2.7. *Let X be a geodesic δ-hyperbolic space and let $Z \subset \partial X$ be compact. Then $\operatorname{Co} Z$ quasi-isometrically embeds into X.*

Proof. Fix a basepoint $x_0 \in X$ and let $\delta' = \delta(\operatorname{Co} Z)$. Since d is visual for both X and $\operatorname{Co} Z$, up to rescaling X we may assume that $(z|z')_{x_0}$ and $(z|z')_o$ are uniformly close for all $z, z' \in Z$. For each $z \in Z$, fix a representative $(\kappa_0, 12\delta)$-quasi-geodesic ray $\rho_z \in z$ eminating from x_0 by Remark 1.6, where $\kappa_0 = 1 + 4\delta$. Let $\iota : \operatorname{Co} Z \to X$ be the map $(z,t) \mapsto \rho_z(t)$.

Since $\gamma_z \in z$ is geodesic, $(z|\gamma_z(t))_o > |\gamma_z(t)| - \delta'$ and $|\gamma_z(t)| = t$. Likewise, since $\rho_z \in z$ is $(\kappa_0, 12\delta)$-quasi-geodesic, $(z|\rho_z(t))_{x_0} > |\rho_z(t)| - M - \delta$, where $M = M(\kappa_0, 12\delta)$ is the Morse constant, and $|\rho_z(t)| = \kappa_z(t)t + O_\delta(1)$ with $\frac{1}{\kappa_0} \leq \kappa_z(t) \leq \kappa_0$. Let $y = (z,t), y' = (z',t') \in \operatorname{Co} Z$. By [BS00, Lem. 5.1], we have

$$\begin{aligned}
|yy'| &= |\gamma_z(t)\gamma_{z'}(t')| \\
&= |\gamma_z(t)| + |\gamma_{z'}(t')| - 2\min\{|\gamma_z(t)|, |\gamma_{z'}(t')|, (z|z')_o\} + O_{\delta'}(1) \\
&= t + t' - 2\min\{t, t', (z|z')_o\} + O_{\delta'}(1)
\end{aligned}$$

and similarly,

$$\begin{aligned}
|\iota(y)\iota(y')| &= |\rho_z(t)\rho_{z'}(t')| \\
&= \kappa_z(t)t + \kappa_{z'}(t')t' - 2\min\{\kappa_z(t)t, \kappa_{z'}(t')t', (z|z')_{x_0}\} + O_\delta(1).
\end{aligned}$$

ι is a quasi-isometric embedding. $\square$

Applying the argument in [BL08, Prop. 6.5], we obtain that $\operatorname{asdim} \operatorname{Co} Z \geq \dim Z + 1$. Proposition 2.5 then follows from Lemma 2.7. //

3. Asymptotic dimension of $\mathcal{G}(\Omega)$

We prove Theorem 1.2. Let Ω be a surface of infinite topological type with finite grand splitting $\mathcal{S}(\Omega)$.

Definition 3.1. An essential, connected, compact subsurface $\Sigma \subset \Omega$ is *fully separating* if every component of $\partial\Sigma$ is separating.

Any compact subsurface can be enlarged to one that is fully separating: *e.g.* we may glue 1-handles between boundary components adjacent to the same complementary component and take the compact surface filled by the result.

Lemma 3.2. *Suppose that $\Sigma \subset \Omega$ is a fully separating non-annular witness for $\mathcal{G}(\Omega)$ and $|\mathcal{S}(\Omega)| = m$. There exists a minimally m-partite relation Γ on $\pi_0(\partial\Sigma)$ such that $\mathcal{A}(\Sigma, \Gamma)$ quasi-isometrically embeds into $\mathcal{G}(\Omega)$. In particular, Γ is not bipartite if $|\mathcal{S}(\Omega)| > 2$.*

Proof. Since Σ is a witness for $\mathcal{G}(\Omega)$, it must separate distinct sets in $\mathcal{S}(\Omega)$. In particular, each boundary component is adjacent to a complementary component containing ends in at most one set in $\mathcal{S}(\Omega)$. Color each component $c \in \pi_0(\partial\Sigma)$ with the corresponding set $e(c) \in \mathcal{S}(\Omega)$, if one exists; let Γ be the complete m-partite relation on these colors (components without a corresponding class are left isolated).

Fix a hyperbolic metric on Σ. For each colored boundary component c, choose a parameterization $c : [0, 1) \to \Sigma$ and a simple ray ρ_c disjoint from $\operatorname{int}(\Sigma)$ with origin $c(0)$ and converging to an end in $e(c)$. Let $a \in \mathcal{A}(\Sigma, \Gamma)$ be an arc that terminates on $c_1, c_2 \in \pi_0(\partial\Sigma)$. Let α be the geodesic representative for a with endpoints $c_i(t_i)$ and define $\delta_i = c_i|_{[0,t_i]}$ to be the subpath of c_i between $c_i(0)$ and $c_i(t_i)$. Let $\alpha^\dagger$ denote the extension of α from both endpoints by $\bar{\delta}_i * \rho_{c_i}$, for $i = 1, 2$ as appropriate. $\alpha^\dagger$ is a simple arc converging to ends in $e(c_1), e(c_2)$ respectively, which are distinct in $\mathcal{S}(\Omega)$ by our choice of Γ. $\alpha^\dagger$ is a grand arc. The map $a \mapsto [\alpha^\dagger]$ preserves disjointness hence extends to a simplicial (1-Lipschitz) map $\psi : \mathcal{A}(\Sigma, \Gamma) \to \mathcal{G}(\Omega)$.

We show that ψ is a quasi-isometric embedding by constructing a coarse Lipschitz retraction $\pi : \mathcal{G}(\Omega) \to \mathcal{A}(\Sigma, \Gamma)$. For a grand arc $w \in \mathcal{G}(\Omega)$, fix a representative ω that is geodesic in Σ. Let $\omega^\pm$ denote the first and last intersections of ω with Σ and let $\hat{\omega}$ denote the shortest path between ω^- and ω^+ in $(\omega \cap \Sigma) \cup \partial\Sigma$. Since ω converges to maximal ends distinguished by $\mathcal{S}(\Omega)$, $\omega^\pm$ lie on boundary components with distinct colors: isotoping $\hat{\omega}$ into the interior of Σ rel $\omega^\pm$, $\hat{\omega}$ is Γ-allowed and we define $\pi : w \mapsto [\hat{\omega}]$. From the constructions of ψ, π, it is immediate that $\pi\psi$ is identity on $\mathcal{A}(\Sigma, \Gamma)$. We verify that π is Lipschitz. Let $w, w' \in \mathcal{G}(\Omega)$ be disjoint grand arcs and let $\pi(w) = [\hat{\omega}]$ and $\pi(w') = [\hat{\omega}']$ as above. Since $\hat{\omega}$ is constructed as a shortest path, it contains at most $|\pi_0(\partial\Sigma)| - 1$ segments that are components of $\omega \cap \Sigma$. Each of these segments intersects $\hat{\omega}'$ at most twice and in subsegments of $\hat{\omega}'$ parallel to $\partial\Sigma$, and the same statement holds exchanging $\hat{\omega}$ and $\hat{\omega}'$. Thus $i(\hat{\omega}, \hat{\omega}') \leq 4|\pi_0(\partial\Sigma)| - 4$. Finally, since $d([\hat{\omega}], [\hat{\omega}']) \leq i(\hat{\omega}, \hat{\omega}') + 1$ by [Kop23b, Prop. 2.6], we obtain that π is $(4|\pi_0(\partial\Sigma)| - 3)$-Lipschitz. $\square$

Witnesses for $\mathcal{G}(\Sigma)$ exist [BNV22, Lem. 2.7] and their enlargements are likewise witnesses, hence there exist fully separating witnesses $\Sigma \subset \Omega$ of arbitrarily large complexity. If $|\mathcal{S}(\Omega)| > 2$ and Γ is chosen as in Lemma 3.2, then $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic by Theorem 1.5 and by Lemma 3.2 and Theorem 1.1 $\operatorname{asdim} \mathcal{G}(\Omega) > n$ for all n.

Suppose instead that $|\mathcal{S}(\Omega)| = 2$. If Ω has infinite genus or infinitely many non-maximal ends, then there exists an infinite collection of pairwise-disjoint annular witnesses separating the sets $\{e, f\} = \mathcal{S}(\Omega)$. Choosing finite subcollections defines quasi-flats of arbitrarily large dimension [Sch, Exercise 3.13], hence again $\operatorname{asdim} \mathcal{G}(\Omega) = \infty$. Alternatively, $\mathcal{G}(\Omega)$ contains an asymphoric hierarhically hyperbolic space of arbitrarily high rank, hence has infinite asymptotic dimension [Kop23a, Prop. 1.11].

Finally, suppose that $|\mathcal{S}(\Omega)| = 2$ and Ω has finite genus and finitely many non-maximal ends. Ω must have at least one infinite set $e \in \mathcal{S}(\Omega)$; let $f \in \mathcal{S}(\Omega)$ be the other set. For any n, choose a $(n+1)$-holed sphere $\Sigma \subset \Omega$ with n boundary components partitioning e and the remaining component separating e from f and any genus or non-maximal ends. Then Σ is a fully separating witness for $\mathcal{G}(\Omega)$ and Γ, defined as in Lemma 3.2, is a n-pointed star. $\mathcal{A}(\Sigma, \Gamma)$ is δ-hyperbolic by Theorem 1.5: we conclude by Lemma 3.2 and Theorem 1.1. //

4. Asymptotic dimension of arbitrary combinatorial models

We generalize the preceding arguments to a broad class of combinatorial models for finite and infinite-type surfaces.

4.1. Admissible combinatorial models.

Let Ω be a orientable surface of finite or infinite type. We first provide an extension of arc and curve systems and markings on Ω that subsumes both.

Definition 4.1. A *cleanly marked arc and curve system* ω on Ω is the union of an arc system α and a marking μ on Ω such that:

(i) $\alpha, \operatorname{base} \mu$ are disjoint, and
(ii) the maximal submarking μ' with only non-empty transversals is complete and clean in each component of $\Omega \setminus (\omega \setminus \mu')$ that it intersects.

A marking which satisfies the above is *locally clean.*

Let $\operatorname{base}\omega = \alpha \cup \operatorname{base}\mu$ denote the underlying arc and curve system. For a component $a \in \operatorname{base}\omega$, let $\operatorname{trans}_\omega(a) := \operatorname{trans}_\mu(a)$ if $a \in \operatorname{base}\mu$, else $\varnothing$ if $a \in \alpha$. We define the geometric intersection of $\omega = \alpha \cup \mu$ and $\omega' = \alpha' \cup \mu'$ to be

$$i(\omega, \omega') := i(\operatorname{base}\omega, \operatorname{base}\omega') + \sum_{a \in \operatorname{base}\mu \cup \operatorname{base}\mu'} \operatorname{diam}_{\mathcal{C}(a)}(\pi_a \omega \cup \pi_a \omega')$$

where $\pi_a(\omega) := \pi_a\alpha \cup \pi_a\mu$ and likewise for ω'. An essential, non-pants subsurface intersects ω if and only if it intersects α or μ.

Notation. Let $\mathcal{MS}(\Omega)$ denote the set of cleanly marked arc and curve systems on Ω.

Definition 4.2. A *multiarc and curve graph* $\mathcal{A}$ on Ω is a simplicial graph whose vertices are arc and curve systems on Ω. Likewise, a *marking graph* $\mathcal{M}$ on Ω is a simplicial graph whose vertices are locally clean markings. Most generally, a *combinatorial model* for Ω is a simplicial graph whose vertices are cleanly marked arc and curve systems on Ω.

4.1.1. *Witness projections.* Let $\Sigma \subset \Omega$ be a compact, essential, non-pants, non-annular subsurface. Let $\mathcal{MS}(\Omega, \Sigma) \subset \mathcal{MS}(\Omega)$ denote the subset of cleanly marked arc and curve systems intersecting Σ. We construct a projection $\rho_\Sigma : \mathcal{MS}(\Omega, \Sigma) \to \mathcal{MS}(\Sigma)$ as follows (see *e.g.* [Sch, §5.2]). Let $\iota : \Sigma \hookrightarrow \Omega$ be the inclusion map, let $p : \Omega_\Sigma \to \Omega$ be the covering space associated to $\pi_1(\Sigma) \cong \operatorname{im}\iota_* < \pi_1(\Omega)$ with Gromov closure $\overline{\Omega}_\Sigma$. Let $\tilde\iota : \Sigma \hookrightarrow \Omega_\Sigma$ be the (unique) lift of ι, and $\bar\iota$ its inclusion into $\overline{\Omega}_\Sigma$. Fix any homeomorphism $\sigma : \overline{\Omega}_\Sigma \to \Sigma$ that is a homotopy inverse for $\bar\iota$; note that σ is unique up to homotopy, hence isotopy.

$$\begin{tikzcd}
 & \Omega_\Sigma \arrow[r, hook] \arrow[d, "p"] & \overline{\Omega}_\Sigma \arrow[dll, bend right, "\sigma"'] \\
\Sigma \arrow[r, hook, "\iota"'] \arrow[ur, hook, "\tilde\iota"] & \Omega &
\end{tikzcd}$$

Given $\omega \in \mathcal{MS}(\Omega, \Sigma)$, let $\rho_\Sigma(\omega)$ be the isotopy class defined by the closures of non-peripheral components of $\sigma p^{-1}(\omega)$. In particular, ρ_Σ preserves (only) essential curves in Σ: if $a \in \operatorname{base}\rho_\Sigma(\omega)$ is a curve component then $a \in \omega$ and a is essential in Σ; if $\operatorname{trans}_\omega(a) = \pi_a b$ then likewise $b \subset \Sigma$ since ω is cleanly marked and we again assign transversal $\pi_a b$.

One verifies that $\rho_\Sigma(\omega)$ is cleanly marked and independent of the choice of representative for ω and σ. Likewise, ρ_Σ is independent of the choice of embedding of Σ: if $\iota' : \Sigma \hookrightarrow \Omega$ is isotopic to ι, then the lift $\bar{\iota}'$ is isotopic to $\bar{\iota}$ and thus a homotopy inverse for σ.

The natural action of $\mathrm{PMod}(\Sigma)$ on $\mathcal{MS}(\Sigma)$ defines an action of $\mathrm{Mod}(\Sigma, \partial\Sigma) \twoheadrightarrow \mathrm{PMod}(\Sigma)$. Similarly, $\mathrm{Mod}(\Sigma, \partial\Sigma) \curvearrowright \mathcal{MS}(\Omega, \Sigma)$ via the homomorphism $\mathrm{Mod}(\Sigma, \partial\Sigma) \to \mathrm{PMod}(\Omega)$ obtained by extending by identity.

Lemma 4.3. $\rho_\Sigma : \mathcal{MS}(\Omega, \Sigma) \to \mathcal{MS}(\Sigma)$ *is* $\mathrm{Mod}(\Sigma, \partial\Sigma)$*-equivariant.*

Proof. Let $\varphi_0 \in \mathrm{Mod}(\Sigma, \partial\Sigma)$, fixing a representative. Let $\varphi \in \mathrm{PMod}(\Omega)$ be its extension by identity; since φ is (compactly) supported in Σ, it lifts to a quasi-isometry on Ω_Σ that extends to a homeomorphism $\overline{\varphi}$ on $\overline{\Omega}_\Sigma$. Since $\bar{\iota}\varphi_0 = \overline{\varphi}\,\bar{\iota}$ and $\sigma, \bar{\iota}$ are homotopy inverses, $\varphi_0\sigma$ and $\sigma\overline{\varphi}$ are homotopic and thus isotopic. For $\omega \in \mathcal{MS}(\Omega, \Sigma)$, $\sigma\overline{p^{-1}(\varphi\omega)} = \sigma\overline{\varphi}\,\overline{p^{-1}(\omega)}$ is isotopic to $\varphi_0\sigma\overline{p^{-1}(\omega)}$, whence the claim follows. □

Corollary 4.4. *Let* $\phi \in \mathrm{PMod}(\Sigma)$*. Then there exists* $\psi \in \mathrm{PMod}(\Omega)$ *preserving* $\mathcal{MS}(\Omega, \Sigma)$ *such that for any* $\omega \in \mathcal{MS}(\Omega, \Sigma)$*,* $\phi\rho_\Sigma(\omega) = \rho_\Sigma(\psi\omega)$*.* □

Given a combinatorial model $\mathcal{M}$ on Ω, let $V(\mathcal{M}), E(\mathcal{M})$ denote its vertex and edge sets, respectively. If Σ is a witness for $\mathcal{M}$ then $V(\mathcal{M}) \subset \mathcal{MS}(\Omega, \Sigma)$ and ρ_Σ defines a projection $V(\mathcal{M}) \to \mathcal{MS}(\Sigma)$.

Definition 4.5. A connected combinatorial model $\mathcal{M}$ on Ω is *admissible* if

(i) $\mathcal{M}$ admits a (compact) witness,
(ii) $\mathrm{PMod}(\Omega)$ preserves $V(\mathcal{M})$ and extends to an action on $\mathcal{M}$, and
(iii) for any non-annular witness $\Delta \subset \Omega$, there exists L_Δ such that if $(a, b) \in E(\mathcal{M})$, then $i(\rho_\Delta(a), \rho_\Delta(b)) \leq L_\Delta$.

Remark 4.6. When Ω is finite-type, it deformation retracts to a compact witness $\overline{\Omega}$. Since in addition $i(\rho_\Delta(a), \rho_\Delta(b)) \leq i(a, b) = i(\rho_{\overline{\Omega}}(a), \rho_{\overline{\Omega}}(b))$, (i) is tautological and in (iii) we may choose $L_\Delta = L_{\overline{\Omega}}$ to be uniform.

Admissible combinatorial models include many familiar graphs, including the curve graph $\mathcal{C}\Omega$, the 1-skeleton $\mathcal{MC}(\Omega)^{(1)}$ of Masur–Minsky's marking complex, and the prescribed arc graphs and grand arc graph discussed above.

4.1.2. *Combinatorial models on witnesses.* Let $\Sigma \subset \Omega$ be a non-annular witness for an admissible combinatorial model $\mathcal{M}$ on Ω. We construct an admissible combinatorial model $\mathcal{M}_\Sigma$ on Σ for which the projection ρ_Σ restricts to a Lipschitz map $\mathcal{M} \to \mathcal{M}_\Sigma$, along with a Lipschitz coarse section $\iota : \mathcal{M}_\Sigma \to \mathcal{M}$. It follows that $\mathcal{M}_\Sigma$ quasi-isometrically embeds into $\mathcal{M}$.

Let $V(\mathcal{M}_\Sigma) = \rho_\Sigma(V(\mathcal{M}))$ and let $(a, b) \in E(\mathcal{M}_\Sigma)$ if and only if $a \neq b$ and there exist $\tilde{a} \in \rho_\Sigma^{-1}(a), \tilde{b} \in \rho_\Sigma^{-1}(b)$ such that $(\tilde{a}, \tilde{b}) \in E(\mathcal{M})$. It is immediate that $\rho_\Sigma : V(\mathcal{M}) \to V(\mathcal{M}_\Sigma)$ extends to a surjective 1-Lipschitz map $\rho_\Sigma : \mathcal{M} \to \mathcal{M}_\Sigma$, hence in particular since $\mathcal{M}$ is connected so is $\mathcal{M}_\Sigma$. Likewise, since $\mathcal{M}$ satisfies Definition 4.5(iii), so does $\mathcal{M}_\Sigma$ for uniform $L = L_\Sigma$. By Corollary 4.4 $\mathrm{PMod}(\Sigma)$ acts naturally on $\mathcal{M}_\Sigma$, hence $\mathcal{M}_\Sigma$ is admissible.

Fix any $\mathrm{Mod}(\Sigma, \partial\Sigma)$-equivariant section $\iota : V(\mathcal{M}_\Sigma) \to V(\mathcal{M})$, and let $\tilde{a} = \iota(a) \in \rho_\Sigma^{-1}(a)$. We show that ι is Lipschitz, hence extends to a Lipschitz coarse section $\iota : \mathcal{M}_\Sigma \to \mathcal{M}$ for ρ_Σ. Since for any $(a,b) \in E(\mathcal{M}_\Sigma)$, $i(a,b) \leq L$, there are finitely many $\mathrm{Mod}(\Sigma, \partial\Sigma)$-orbits of edges in $\mathcal{M}_\Sigma$. Let

$$M = \max_{(a,b)\in E(\mathcal{M}_\Sigma)/G} d_{\mathcal{M}}(\tilde{a}, \tilde{b})$$

where $G = \mathrm{Mod}(\Sigma, \partial\Sigma)$. Then ι is M-Lipschitz. We have shown:

Proposition 4.7. *Let $\Sigma \subset \Omega$ be a non-annular witness for an admissible combinatorial model $\mathcal{M}$ on Ω. There exists an admissible combinatorial model $\mathcal{M}_\Sigma$ on Σ which quasi-isometrically embeds into $\mathcal{M}$.* □

4.2. **Asymptotic dimension lower bounds.** We first consider the asymptotic dimension of admissible combinatorial models on Σ, a finite-type surface. Up to deformation retraction, we assume Σ is compact.

Remark 4.8. If Σ is a (closed) torus, then any admissible combinatorial model is quasi-isometric to the curve graph, hence a quasi-tree with asdim = 1. Otherwise, if Σ admits a non-empty admissible combinatorial model (and in particular, a witness subsurface), then $\chi(\Sigma) \leq -1$ and $\Sigma \ncong \Sigma_0^3$.

4.2.1. *Marking graphs and rank.* It suffices to consider admissible marking graphs, in the sense of the following lemma:

Lemma 4.9. *Let $\mathcal{M}$ be an admissible combinatorial model on a compact surface $\Sigma \ncong \Sigma_1$. Then there exists an admissible marking graph $\mathcal{M}'$ on Σ with an identical witness set and a* $\mathrm{PMod}(\Sigma)$*-equivariant quasi-isometry $\mathcal{M} \to \mathcal{M}'$ which coarsely preserves witness subsurface projection.*

For a simple closed curve $a \subset \Sigma$, let $d_a(\omega, \omega') := \mathrm{diam}_{\mathcal{C}(a)}(\pi_a\omega \cup \pi_a\omega')$. Then $d_a(\omega, \omega')$ is bounded in terms of $i(\omega, \omega')$ uniformly in a (see [Wat16, Thm. 2.10], *e.g.*).

Proof. Let $\omega = \nu \cup \alpha \in V(\mathcal{M})$. From [Kop23a, §3], we construct a canonical set of locally clean markings $\mu_{\mathrm{base}\,\omega}$ corresponding to the arc and curve system $\mathrm{base}\,\omega$ such that

- *(i)* for $\mu \in \mu_{\mathrm{base}\,\omega}$, $\mathrm{base}\,\nu \subset \mu$ with empty transversals, and
- *(ii)* an essential, non-pants subsurface intersects $\mu \in \mu_{\mathrm{base}\,\omega}$ if and only if it intersects $\mathrm{base}\,\omega$.

Moreover, there exists M such that for any $\omega, \omega' \in V(\mathcal{M})$

- *(iii)* $i(\omega, \mu) < M$ for $\mu \in \mu_{\mathrm{base}\,\omega}$,[1] and

[1] This fact is not stated in [Kop23a], but follows from the construction of $\mu_{\mathrm{base}\,\omega}$.

(iv) if $(\omega, \omega') \in E(\mathcal{M})$ then $i(\mu, \mu') < M$ for any $\mu \in \mu_{\mathrm{base}\,\omega}, \mu' \in \mu_{\mathrm{base}\,\omega'}$.

Obtain the set μ_ω by adding the transversals in ν to each $\mu \in \mu_{\mathrm{base}\,\omega}$. Note that μ remains locally clean: else, there is a component in ν bounding an essential, non-pants subsurface disjoint from μ but not from $\mathrm{base}\,\omega$, contradicting property *(ii)*. Likewise *(iii)* still holds for μ_ω.

Let $V(\mathcal{M}') = \bigcup_{\omega \in \mathcal{M}} \mu_\omega$ and let $(\mu, \mu') \in E(\mathcal{M}')$ if and only if $\mu \in \mu_\omega, \mu' \in \mu_{\omega'}$ for some $(\omega, \omega') \in E(\mathcal{M})$. We prove that $\mathcal{M}'$ is admissible and $\omega \mapsto \mu_\omega$ is the desired (coarse) quasi-isometry. By applying the arguments in [Kop23a] it suffices to verify *(ii)* and *(iv)* for the sets μ_ω, replacing $\mathrm{base}\,\omega$ with ω and $\mathrm{base}\,\omega'$ with ω', as well as the property

(v) if $\mu_\omega \cap \mu_{\omega'} \neq \varnothing$, then $i(\omega, \omega')$ is uniformly bounded.

(ii) for $\mu_{\mathrm{base}\,\omega}$ implies the same for μ_ω, except for annuli parallel to curves in $\mathrm{base}\,\nu$; since $\nu \subset \mu \cap \omega$ for $\mu \in \mu_\omega$, *(ii)* holds for μ_ω. Let $\omega = \nu \cup \alpha$ and $\omega' = \nu' \cup \alpha'$. Suppose $(\omega, \omega') \in E(\mathcal{M})$ and $\mu \in \mu_\omega, \mu' \in \mu_{\omega'}$. Since *(iv)* holds for $\mu_{\mathrm{base}\,\omega}, \mu_{\mathrm{base}\,\omega'}$, it suffices that $d_a(\mu, \mu')$ is uniformly bounded for $a \in \nu \cup \nu'$; by *(ii)*, if μ projects to $\mathcal{C}(a)$ then so does ω, and likewise for μ' and ω'. $i(\mu, \omega), i(\mu', \omega') < M$, hence each pair has uniformly close projections if non-empty: $d_a(\mu, \mu')$ is bounded in terms of $d_a(\omega, \omega') < L_\Sigma$.

Finally, if $\mu \in \mu_\omega \cap \mu_{\omega'}$, then [Kop23a, §3.2] implies that $i(\mathrm{base}\,\omega, \mathrm{base}\,\omega')$ is uniformly bounded. For any $a \in \nu \cup \nu'$, we show that $d_a(\omega, \omega')$ is also uniformly bounded. By construction, if ω or ω' has non-empty projection to $\mathcal{C}(a)$ then so does μ. Since $i(\omega, \mu), i(\omega', \mu) < M$, we conclude as above. □

In particular, by [Kop23a] any admissible marking graph (hence likewise any admissible combinatorial model) $\mathcal{M}$ on a compact surface Σ is an asymphoric hierarchically hyperbolic space with respect to subsurface projections to witness curve graphs. Let $\mathscr{X}$ denote the collection of witness subsurfaces for $\mathcal{M}$. Then in particular the *rank* ν of $(\mathcal{M}, \mathscr{X})$ corresponds to the largest cardinality of a set of pairwise disjoint, connected surfaces in $\mathscr{X}$. Since $(\mathcal{M}, \mathscr{X})$ is asymphoric, $\mathrm{asdim}\,\mathcal{M} \geq \nu$ [BHS21, Thm. 1.15] and $\mathcal{M}$ is δ-hyperbolic if and only if $\nu = 1$ [BHS21, Cor. 2.15]. The lower bound here will prove sufficient except when $\nu = 1$; we note that an identical bound can be achieved by explicitly constructing quasi-flats.

4.2.2. *The δ-hyperbolic case.* Adapting the arguments in Section 2, we prove the following:

Theorem 4.10. *Let Σ be a genus g compact surface, possibly with boundary. If $\mathcal{M}$ is a (non-empty) δ-hyperbolic admissible combinatorial model on Σ, then* $\mathrm{asdim}\,\mathcal{M} \geq g - \lceil \frac{1}{2}\chi(\Sigma) \rceil$.

If $\Sigma \cong \Sigma_1$, then the claim is immediate by Remark 4.8. Otherwise, we may assume $\mathcal{M}$ is an admissible marking graph by Lemma 4.9. Let $\mathscr{X}^{\mathcal{M}}$ denote the collection of witness subsurfaces for $\mathcal{M}$. For any $\mathcal{M}'$ an admissible marking graph on Σ with $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, there exists a functorial canonical coarse surjection $\iota : \mathcal{M} \to \mathcal{M}'$ such that $\pi_W \circ \iota$ is uniformly coarsely π_W for any $W \in \mathscr{X}^{\mathcal{M}'}$ [Kop23a, §2.1]. In particular, $\mathscr{X}^{\mathcal{MC}^{(1)}(\Sigma)}$ is every essential, non-peripheral subsurface in Σ and $\mathscr{X}^{\mathcal{C}\Sigma} = \{\Sigma\}$, hence we have canonical maps $\mathcal{MC}^{(1)}(\Sigma) \to \mathcal{M} \to \mathcal{C}\Sigma$.

Lemma 4.11. *Let $\mathcal{M}, \mathcal{M}'$ be admissible marking graphs on Σ, a compact surface, such that $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, and let $\iota : \mathcal{M} \to \mathcal{M}'$ be the canonical coarse surjection. If $\mathcal{M}$ is δ-hyperbolic, then ι is coarsely alignment-preserving.*

We note that if $\mathcal{M}$ is δ-hyperbolic, then $\nu(\mathcal{M}') \leq \nu(\mathcal{M}) \leq 1$, hence $\mathcal{M}'$ is δ-hyperbolic. Recall that a path $\rho \subset X$ is a *D-hierarchy path* for a hierarchically hyperbolic space $(X, \mathscr{G})$ if it is a (D, D)-quasi-geodesic and $\pi_\alpha \rho$ is a unparameterized (D, D)-quasi-geodesic for all $\alpha \in \mathscr{G}$.

Proof. Since $(\mathcal{M}, \mathscr{X}^{\mathcal{M}})$ is hierarchically hyperbolic, there exists $D > 0$ such that for any $x, y \in \mathcal{M}$, there exists a D-hierarchy path joining x, y [BHS19, Thm. 4.4]. Let $(x, z, y) \in \mathcal{M}^3$ be aligned and let γ be the geodesic from x to y passing through z and ρ the hierarchy path between x, y. By the Morse lemma, there exists a constant $M(D, \delta)$ such that γ, ρ are $M(D, \delta)$-Hausdorff close, hence $d(z, \rho) \leq M(D, \delta)$. For any $W \in \mathscr{X}^{\mathcal{M}}$, $\pi_W \rho$ is an unparameterized (D, D)-quasi-geodesic. Applying the Morse lemma and that π_W is L-Lipschitz for uniform L, it follows that $(\pi_W(x), \pi_W(z), \pi_W(y))$ are K-aligned where $K = 2(M(D, \delta_0) + LM(D, \delta))$ is uniform over $\mathcal{M}^3$, $\mathscr{X}^{\mathcal{M}}$ and δ_0 is a uniform hyperbolicity constant for curve graphs [HPW15].

It follows that π_W for $W \in \mathscr{X}^{\mathcal{M}}$ and likewise $\pi_{W'}$ for $W' \in \mathscr{X}^{\mathcal{M}'}$ are K'-alignment preserving for uniform K'. Since $\mathscr{X}^{\mathcal{M}'} \subset \mathscr{X}^{\mathcal{M}}$, the distance formulas for $\mathcal{M}, \mathcal{M}'$ imply the claim. □

Suppose that $\mathcal{M}$ is a δ-hyperbolic admissible marking graph on a compact surface Σ with genus g. By Lemma 4.11, the canonical map $\iota : \mathcal{M} \to \mathcal{C}\Sigma$ is coarsely alignment preserving, hence by Theorem 1.12 $\partial\mathcal{C}\Sigma$ embeds into $\partial\mathcal{M}$. To prove Theorem 4.10 it suffices to find a compact subspace $Z \subset \partial\mathcal{C}\Sigma$ such that $\dim Z \geq n := g - 1 - \lceil \frac{1}{2}\chi(\Sigma) \rceil$, since by Proposition 2.5 $\dim Z + 1 \leq \operatorname{asdim} \mathcal{M}$. Recall that $\partial\mathcal{C}\Sigma \cong \mathcal{EL}(\Sigma)$.

Proposition 4.12. *Let Σ be a genus g compact hyperbolic surface and S the $(n+4)$-times punctured sphere, where $n = g - 1 - \lceil \frac{1}{2}\chi(\Sigma) \rceil$. Then $\mathcal{EL}(S)$ embeds into $\partial\mathcal{C}\Sigma \cong \mathcal{EL}(\Sigma)$.*

Proof. For simplicity, we replace the boundary components of Σ with punctures, noting that $\mathcal{C}\Sigma \cong \mathcal{C}(\Sigma \setminus \partial\Sigma)$. Choose a hyperelliptic involution η on Σ that fixes at most one puncture and let $h : \Sigma \to S'$ be the orbifold covering map obtained by quotienting by η. Obtain S by removing the cone points of S': one verifies that S has $n+4$ punctures. By [RS09], h induces a quasi-isometric embedding $h_* : \mathcal{C}S \to \mathcal{C}\Sigma$, which has quasi-convex image by the Morse lemma. Hence $\mathcal{EL}(S) \cong \partial\mathcal{C}S \subset \partial\mathcal{C}\Sigma$. □

When Σ is a sphere with four boundary components, Theorem 4.10 is vacuously true. Otherwise, from Theorem 2.4 and the universal embedding property of Nöbeling spaces, we obtain the desired subspace $Z \subset \mathcal{EL}(S) \subset \partial\mathcal{C}\Sigma$ and Theorem 4.10 follows. //

4.2.3. *Lower bounds for infinite-type surfaces.* Given an admissible combinatorial model $\mathcal{M}$ on an infinite-type surface Ω, let $w_{\mathcal{M}} \in \mathbb{N} \cup \{\infty\}$ denote the least upper bound on cardinalities for a set of pairwise-disjoint connected witnesses for $\mathcal{M}$. We consider two cases:

(i) *$w_{\mathcal{M}}$ is infinite.* For arbitrarily large $m \in \mathbb{N}$, we may choose a compact, essential subsurface $\Sigma \subset \Omega$ containing at least m disjoint witnesses. Σ is a witness for $\mathcal{M}$, and any witness for $\mathcal{M}$ contained in Σ is a witness for $\mathcal{M}_\Sigma$ by construction. It follows that $\mathcal{M}_\Sigma$ is an asymphoric hierarchically hyperbolic space of rank $\nu \geq m$, hence by Proposition 4.7 $\operatorname{asdim}\mathcal{M} \geq \operatorname{asdim}\mathcal{M}_\Sigma \geq m$. $\operatorname{asdim}\mathcal{M} = \infty$.

(ii) *$w_{\mathcal{M}} = m$ is finite.* Fix a collection of pairwise disjoint witnesses $\{W_i\}$ with cardinality m. Fix W_0 among these such that W_0 lies in a complementary component Ω_0 of $\bigcup_{i>0} W_i$ of infinite type. Let $\Sigma \subset \Omega_0$ be an enlargement of W_0 of arbitrarily negative $\chi(\Sigma)$: Σ is a witness for $\mathcal{M}$. Moreover, since any witness for $\mathcal{M}_\Sigma$ is a witness for $\mathcal{M}$ disjoint from the $W_{i>0}$, any two connected witnesses for $\mathcal{M}_\Sigma$ must intersect: $\mathcal{M}_\Sigma$ is an asymphoric hierarchically hyperbolic space of rank $\nu = 1$, hence δ-hyperbolic. By Proposition 4.7 and Theorem 4.10, $\operatorname{asdim}\mathcal{M} \geq \operatorname{asdim}\mathcal{M}_\Sigma \geq -\frac{1}{2}\chi(\Sigma)$, hence $\operatorname{asdim}\mathcal{M} = \infty$.

Theorem 4.13. *Let $\mathcal{M}$ be an admissible combinatorial model on an infinite-type surface Ω. Then* $\operatorname{asdim}\mathcal{M} = \infty$. □

Theorem 1.3 follows from Theorems 4.10 and 4.13 in the special case of admissible multiarc and curve graphs. //

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References

[AV20] Javier Aramayona and Nicholas G. Vlamis, *Big mapping class groups: an overview*, 2020.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Springer Berlin, Heidelberg, 1999.

[BHS19] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, *Hierarchically hyperbolic spaces II: Combination theorems and the distance formula*, Pacific Journal of Mathematics **299** (2019), no. 2, 257–338.

[BHS21] ———, *Quasiflats in hierarchically hyperbolic spaces*, Duke Mathematical Journal **170** (2021), no. 5, 909 – 996.

[BL08] Sergei Buyalo and Nina Lebedeva, *Dimensions of locally and asymptotically self-similar spaces*, St. Petersburg Math. J. **19** (2008), 45–65.

[BNV22] Assaf Bar-Natan and Yvon Verberne, *The grand arc graph*, 2022.

[BS00] Mario Bonk and Oded Schramm, *Embeddings of Gromov hyperbolic spaces*, Geometric and Functional Analysis **10** (2000), 243–284.

[Buy06] Sergei Buyalo, *Asymptotic dimension of a hyperbolic space and capacity dimension of its boundary at infinity*, St. Petersberg Math. J. **17** (2006), 267–283.

[Cal09] Danny Calegari, *Big mapping class groups and dynamics*, Blog post, `https://lamington.wordpress.com/2009/06/22/big-mapping-class-groups-and-dynamics`, 2009.

[CPV21] Yassin Chandran, Priyam Patel, and Nicholas G. Vlamis, *Infinite type surfaces and mapping class groups: Open problems*, `http://qcpages.qc.cuny.edu/~nvlamis/Papers/InfTypeProblems.pdf`, 2021.

[DK18] C. Druţu and M. Kapovich, *Geometric group theory*, Colloquium Publications, American Mathematical Society, 2018.

[DT17] Spencer Dowdall and Samuel J. Taylor, *The co-surface graph and the geometry of hyperbolic free group extensions*, Journal of Topology **10** (2017), no. 2, 447–482.

[FGM21] Federica Fanoni, Tyrone Ghaswala, and Alan McLeay, *Homeomorphic subsurfaces and the omnipresent arcs*, Annales Henri Lebesgue **4** (2021), 1565–1593.

[Gab14] David Gabai, *On the topology of ending lamination space*, Geometry & Topology **18** (2014), no. 5, 2683 – 2745.

[Has22] Yo Hasegawa, *Gromov boundaries of non-proper hyperbolic geodesic spaces*, Tokyo Journal of Mathematics **45** (2022), no. 2, 319–331.

[HPW15] Sebastian Hensel, Piotr Przytycki, and Richard Webb, *1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs*, Journal of the European Mathematical Society **17** (2015), no. 4, 755–762.

[KB02] Ilya Kapovich and Nadia Benakli, *Boundaries of hyperbolic groups*, 2002.

[Kla18] Erica Klarreich, *The boundary at infinity of the curve complex and the relative teichm uller space*, Groups, Geometry, and Dynamics **16** (2018).

[Kop23a] Michael C. Kopreski, *Multiarc and curve graphs are hierarchically hyperbolic*, 2023.

[Kop23b] ———, *Prescribed arc graphs*, 2023.

[MM00] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geometric & Functional Analysis **10** (2000), 902–974.

[MR20] Kathryn Mann and Kasra Rafi, *Large scale geometry of big mapping class groups*, 2020.

[Nöb30] G. Nöbeling, *Über eine n-dimensionale Universalmenge im* $\mathbb{R}_{2n+1}$, Mathematische Annalen **104** (1930), 71–80.

[Po17] Witsarut Pho-on, *Infinite unicorn paths and Gromov boundaries*, Groups, Geometry, and Dynamics **11** (2017), no. 1, 353–370.

[Ric63] I. Richards, *On the classification of noncompact surfaces*, Transactions of the American Mathematical Society **106** (1963), 259–269.

[Ros14] Christian Rosendal, *Large scale geometry of metrisable groups*, 2014.

[RS09] Kasra Rafi and Saul Schleimer, *Covers and the curve complex*, Geometry & Topology **13** (2009), no. 4, 2141 – 2162.

[Sch] Saul Schleimer, *Notes on the complex of curves*, `http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf`.

[Wat16] Yohsuke Watanabe, *Intersection numbers in the curve graph with a uniform constant*, Topology and its Applications **204** (2016), 157–167.