MULTIARC AND CURVE GRAPHS ARE HIERARCHICALLY HYPERBOLIC

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Abstract. A multiarc and curve graph is a simplicial graph whose vertices are arc and curve systems on a compact, connected, orientable surface Σ. We show that all connected multiarc and curve graphs preserved cocompactly by the natural action of $\mathrm{PMod}(\Sigma)$ are hierarchically hyperbolic with respect to witness subsurface projection. Here, cocompactness is equivalent to a uniform bound on the geometric intersection number of adjacent vertices. This result extends work of Kate Vokes on twist-free multicurve graphs and confirms two conjectures of Saul Schleimer in a broad setting. In addition, we prove that the coarsely $\mathrm{PMod}(\Sigma)$-equivariant quasi-isometry type of such a graph is uniquely classified by its set of connected witness subsurfaces.

1. Introduction and main results

Let Σ be a compact, connected, orientable, non-pants surface possibly with boundary such that $\chi(\Sigma) \leq -1$. We consider a broad class of natural combinatorial models built from collections of arc and curve systems on Σ:

Definition 1.1. A *multiarc and curve graph* $\mathcal{A}$ on Σ is a simplicial graph whose vertices are collections of disjoint isotopy classes of simple, essential arcs and curves in Σ. $\mathcal{A}$ is a *multicurve graph* if $V(\mathcal{A})$ consists of only multicurves.

Definition 1.2. A connected multiarc and curve graph $\mathcal{A}$ on Σ is *cocompact* if $\mathrm{PMod}(\Sigma)$ preserves $V(\mathcal{A})$ and extends to a cocompact action on $\mathcal{A}$.

A *witness subsurface* W for $\mathcal{A}$ is an essential subsurface of which every component is non-pants and intersects every vertex of $\mathcal{A}$. We prove the following:

Theorem 1.3. *Let $\mathcal{A}$ be a cocompact multiarc and curve graph on Σ and let $\mathscr{X}$ denote its collection of witnesses. Then $(\mathcal{A}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

Theorem 1.4. *The coarsely* $\mathrm{PMod}(\Sigma)$*-equivariant quasi-isometry type of a cocompact multiarc and curve graph on Σ is uniquely determined by its connected witnesses.*

We first show the same for *partial marking graphs*, whose vertex set consists of markings in the sense of [MM00] that are "locally" complete and clean, in Section 3. Admissible partial marking graphs include Masur and Minsky's marking complex $\mathcal{MC}(\Sigma)$. Here, Theorem 1.4 defines a bijection: any set of essential, non-pants

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subsurfaces closed under enlargement and the action of $\mathrm{PMod}(\Sigma)$ is the witness set for some cocompact partial marking graph. In Section 4, we show that any cocompact multiarc and curve graph on Σ is $\mathrm{PMod}(\Sigma)$-equivariantly quasi-isometric to a cocompact partial marking graph on Σ, via a map coarsely preserving subsurface projection, whence Theorems 1.3 and 1.4 follow.

We emphasize that cocompactness is central to our proof of Theorem 1.4. In particular, two cocompact connected graphs are equivariantly quasi-isometric if and only if the vertex stabilizers of each have bounded orbits on the other (see Section 2.5). For multiarc and curve graphs, we show this stabilizer condition is satisfied exactly when two connected witness sets agree.

Finally, we note that cocompactness is equivalent to an explicit criterion on intersection number. Observe the $\mathrm{PMod}(\Sigma)$-action in Definition 1.2 is cocompact if and only if it has finitely many orbits of edges, if and only if the edge geometric intersection numbers $\{i(a,b) : (a,b) \in E(\mathcal{A})\}$ are bounded. Hence equivalently:

Definition 1.5. A connected multiarc and curve graph $\mathcal{A}$ on Σ is *cocompact* if

(i) $\mathrm{PMod}(\Sigma)$ preserves $V(\mathcal{A})$ and extends to an action on $\mathcal{A}$, and
(ii) if $(a,b) \in E(\mathcal{A})$, then the geometric intersection $i(a,b)$ is uniformly bounded.

The present work is indebted to the excellent paper by Kate Vokes on the hierarchical hyperbolicity of *twist-free* multicurve graphs [Vok22], from which our results borrow both inspiration and overall strategy.

1.1. **Applications.** The hierarchical hyperbolicity of cocompact multiarc and curve graphs implies important features of their geometry and that of analogous objects for infinite-type surfaces.

1.1.1. *Two conjectures of Schleimer.* We consider two conjectures of Saul Schleimer [Sch, §2.3] on the geometry of multiarc and curve graphs: the first characterizes δ-hyperbolicity, and the second proposes a distance formula in the sense of [MM00]. Both are resolved positively in the cocompact case by Theorem 1.3.

Given a multiarc and curve graph $\mathcal{A}$ on Σ, let $\pi_W : \mathcal{A} \to 2^{\mathcal{C}W} \setminus \{\varnothing\}$ denote the subsurface projection for a witness $W \subset \Sigma$. For $a, b \in V(\mathcal{A})$, let $d_W(a,b) := \mathrm{diam}_{\mathcal{C}W}(\pi_W(a) \cup \pi_W(b))$.

Conjecture 1.6 (Schleimer)**.** *Let $M \geq 0$ and $\mathcal{A}$ be a multiarc and curve graph such that $(a,b) \in E(\mathcal{A})$ if and only if $i(a,b) \leq M$. $\mathcal{A}$ is δ-hyperbolic if and only if it does not admit disjoint connected witnesses.*

Conjecture 1.7 (Schleimer)**.** *Let $M \geq 0$ and $\mathcal{A}$ be a multiarc and curve graph such that $(a,b) \in E(\mathcal{A})$ if and only if $i(a,b) \leq M$, and let $\mathscr{X}$ denote the collection of witnesses of $\mathcal{A}$. Then there exists $C' = C'(\mathcal{A})$ such that for any $C > C'$, there are constants $K, E \geq 0$ such that*

$$d_{\mathcal{A}}(a,b) \stackrel{K,E}{=} \sum_{W \in \mathscr{X}} [d_W(a,b)]_C \, .$$

If a multiarc and curve graph $\mathcal{A}$ on Σ satisfies (i) of Definition 1.5, then we will say that $\mathcal{A}$ has a *natural* $\mathrm{PMod}(\Sigma)$ action. We note that Schleimer does not assume a natural action of $\mathrm{PMod}(\Sigma)$, and indeed there exist interesting complexes lacking such an action: *e.g.*, the *disk complex* [MS10] and *admissible curve graph* [CR24]. Nonetheless, assuming a natural $\mathrm{PMod}(\Sigma)$ action and the connectivity of $\mathcal{A}$ (as we will for the remaining work), we observe that it is equivalent in the conjectures above to require only that $i(a, b)$ be uniformly bounded for $(a, b) \in E(\mathcal{A})$. In fact:

Remark 1.8. Let $\mathcal{A}$ be a connected multiarc and curve graph with a natural $\mathrm{PMod}(\Sigma)$ action, and suppose $\mathcal{A}'$ is obtained by adding edges between vertices a, b such that $i(a, b) < M$. Then $\mathcal{A} \underset{\text{q.i.}}{\cong} \mathcal{A}'$.

In particular, $\mathrm{id} : V(\mathcal{A}) \to V(\mathcal{A}')$ is bilipschitz. For the lower Lipschitz bound, note that there are finitely many $\mathrm{PMod}(\Sigma)$-orbits of vertex pairs $(a, b) \in E(\mathcal{A}') \setminus E(\mathcal{A})$ since $i(a, b) < M$, hence $d_{\mathcal{A}}(a, b)$ is uniformly bounded.

Conjectures 1.6 and 1.7 follow from the hierarchical hyperbolicity of $\mathcal{A}$ with the usual witness subsurface projection structure. We introduce hierarchical hyperbolicity and make precise these arguments in Section 2.3.

1.1.2. *Cocompact subgraphs of the grand arc graph.* Given Γ a relation on $\pi_0(\partial\Sigma)$, the Γ-*prescribed arc graph* $\mathcal{A}(\Sigma, \Gamma)$, defined by the author in [Kop23], is the full subcomplex of the arc graph $\mathcal{A}(\Sigma)$ spanned by arcs between boundary components in Γ. From Theorem 1.3 it follows that $\mathcal{A}(\Sigma, \Gamma)$ is hierarchically hyperbolic with respect to witness subsurface projection. Moreover, if Γ is bipartite and Σ has genus g and b boundary components, then it may be seen that $\mathcal{A}(\Sigma, \Gamma)$ admits $g + b - 1$ pairwise disjoint annular witnesses: it follows likewise that $\mathcal{A}(\Sigma, \Gamma)$ has rank $\nu \geq g + b - 1$ (see Section 2.3).

As an immediately application, let Ω be an orientable surface of infinite topological type. We consider the *grand arc graph* $\mathcal{G}(\Omega)$, defined in [BNV22], whose vertices are arcs between ends of distinct maximal type in the sense of the partial order given in [MR20]. Suppose that Ω has exactly n distinct maximal end classes. Then for an arbitrarily large witness $W \subset \Omega$ there exists a n-partite relation Γ_W on $\pi_0(\partial W)$ such that, by extending arcs, $\mathcal{A}(W, \Gamma_W)$ quasi-isometrically embeds in $\mathcal{G}(\Omega)$ [Kop24]. Suppose additionally that Ω has exactly $n = 2$ distinct maximal end classes, hence the $\mathcal{A}(W, \Gamma_W)$ are bipartite. Then choosing an exhaustion of Ω by such W, we have the following:

Proposition 1.9. *Suppose that Ω is infinite-type and has exactly 2 distinct maximal end classes. There exists an asymphoric hierarchically hyperbolic arc graph of arbitrarily large rank that quasi-isometrically embeds into $\mathcal{G}(\Omega)$.* □

By [BHS21, Thm. 1.15], it follows that $\operatorname{asdim} \mathcal{G}(\Omega) = \infty$ whenever Ω has exactly two maximal end classes and infinite genus or infinitely many non-maximal ends; we note this fact may be likewise shown by explicitly constructing quasi-flats.

2. Background and preliminaries

Given $\Sigma = \Sigma_g^b$ a compact, connected, orientable surface with genus g and b boundary components, let $\xi(\Sigma) = 3g + b - 3$ denote the *complexity* of Σ, equal to the number of components in a maximal multicurve. Let $\mathcal{AC}(\Sigma)$ denote the *arc and* *curve graph* of Σ, whose vertices are the essential simple arcs and closed curves in Σ up to isotopy, with edges according to disjointness. The *curve graph* $\mathcal{C}\Sigma$ is the full subgraph spanned by curves; the *arc graph* $\mathcal{A}(\Sigma)$ is the full subgraph spanned by arcs. An *arc and curve system* is a collection of isotopy classes of pairwise disjoint, simple, and essential arcs and curves on Σ.

Let $\mathcal{A}$ be a multiarc and curve graph on Σ as above.

Definition 2.1. A compact, essential (π_1-injective, non-peripheral) subsurface $W \subset \Sigma$ is a *witness* for $\mathcal{A}$ if W does not contain a Σ_0^3 component and every arc and curve system in $V(\mathcal{A})$ intersects every component of W.

Note that we do not require witnesses to be connected: if $W = V \sqcup V'$, then we define $\mathcal{C}W$ to be the graph join $\mathcal{C}V * \mathcal{C}V'$ and $\pi_W(a) := \pi_V(a) \cup \pi_{V'}(a)$.

2.1. **Low-complexity cases.** Note that $\xi(\Sigma) > 0$ by assumption. When $\xi(\Sigma) \leq 0$, any multiarc and curve graph on Σ is empty or it is finite and Σ admits no essential (non-peripheral) non-pants subsurfaces, except when $\Sigma \cong \Sigma_1^0$. In the latter case, any cocompact multiarc and curve graph is quasi-isometric to the curve graph and Σ_1^0 is the only witness, hence Theorems 1.3 and 1.4 hold trivially.

2.2. **Twist-free multicurve graphs.** In the sense above, Conjectures 1.6 and 1.7 were resolved positively for a broad class of examples by Kate Vokes in [Vok22].

Definition 2.2 (Vokes)**.** A multicurve graph $\mathcal{G}$ on Σ is *twist-free* if it is cocompact[1] and admits no annular witnesses.

Theorem 2.3 (Vokes)**.** *Let $\mathcal{G}$ be a twist-free multicurve graph. Then $(\mathcal{G}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projections to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

Theorem 1.4 and in fact a bijection analogous to Theorem 3.16 below follow easily from Vokes' results in the special case of twist-free multicurve graphs.

The existence of annular witnesses appears related to non-hyperbolicity, or equivalently by Conjecture 1.6, the existence of disjoint connected witnesses. In particular, if $\xi(\Sigma) > 1$ and a multiarc and curve graph $\mathcal{A}$ on Σ admits an annular witness A, then $\mathcal{A}$ admits a disjoint pair of connected witnesses, namely A and a complementary component. As a partial converse, if $V(\mathcal{A})$ consists of single arcs and curves and $\mathcal{A}$ admits two disjoint witnesses W, W' separated by a unique complementary component, then $\mathcal{A}$ admits an annular witness. For example, the Γ-prescribed arc graph $\mathcal{A}(\Sigma, \Gamma)$ may be viewed as a cocompact multicurve graph by passing to projections to $\mathcal{C}\Sigma$; however, whenever $\Sigma \cong \Sigma_g^b$ and Γ is bipartite (in particular, whenever $\mathcal{A}(\Sigma, \Gamma)$ is non-hyperbolic) $\mathcal{A}(\Sigma, \Gamma)$ admits $g + b - 1$ pairwise disjoint annular witnesses, hence Theorem 2.3 does not apply.

[1] Vokes actually requires a natural action of $\mathrm{Mod}(\Sigma)$; however, any subgroup containing $\mathrm{PMod}(\Sigma)$ suffices for the arguments in [Vok22].

2.3. **Hierarchical hyperbolicity.** Behrstock, Hagen, and Sisto introduced hierarchically hyperbolic spaces in [BHS17] to provide a common geometric framework within which to study mapping class groups and cubical groups. In effect, hierarchical hyperbolicity provides an axiomatization and generalization of the geometric structure of the mapping class groups of finite type surfaces elucidated in [MM00]; we direct the reader to [BHS19] for a full enumeration of the axioms, and to [Sis17] for a non-technical discussion of hierarchical hyperbolicity and a survey of results.

We provide a brief description. Let X be a quasi-geodesic space. A hierarchically hyperbolic structure $(X, \mathcal{G})$ on X is comprised of the following:

- an index set $\mathcal{G}$ and a collection of uniformly δ-hyperbolic spaces $\{\mathcal{C}W : W \in \mathcal{G}\}$ with associated projections $\pi_W : X \to 2^{\mathcal{C}W} \setminus \{\varnothing\}$;
- relations $\sqsubseteq$ (*nesting*) and $\perp$ (*orthogonality*) on $\mathcal{G}$;
- if $U \sqsubseteq V$ then a map $\rho^V_U : \mathcal{C}V \to 2^{\mathcal{C}U}$ and if additionally $U \neq V$, a uniformly bounded set $\rho^U_V \subset \mathcal{C}V$. If not $U \perp V$ and U, V are not $\sqsubseteq$-comparable, then a uniformly bounded set $\rho^U_V \subset \mathcal{C}V$ and *vice versa*.

Let $\pitchfork$ (*transversality*) denote the complement of $\sqsubseteq \cup \perp$; any $U, V \in \mathcal{G}$ are related by exactly one of $\sqsubseteq, \perp, \pitchfork$. In addition, the above must satisfy nine axioms for $(X, \mathcal{G})$ to be hierarchically hyperbolic. We consider the model example: the marking complex $\mathcal{MC}(\Sigma)$ defined in [MM00], on which $\mathrm{Mod}(\Sigma)$ acts geometrically, is hierarchically hyperbolic with respect to the index set $\mathcal{G} = \{\text{essential subsurfaces}\}$ and subsurface projections π_W, with nesting and orthogonality determined by inclusion and disjointness, respectively; ρ^V_U is determined by subsurface projection, or the projection of boundary components, as appropriate [BHS19, Thm. 11.1].

By defining a hierarchically hyperbolic structure analogously for partial marking graphs, many of the axioms follow from the fact that they hold for $\mathcal{MC}(\Sigma)$. We omit these here and enumerate three that we will check explicitly. For $W \in \mathcal{G}$, let $\mathcal{G}^*_W := \{T \in \mathcal{G} : W \neq T \sqsubseteq W\}$ and $d_W(u, v) := \mathrm{diam}_{\mathcal{C}W}(\pi_W(u) \cup \pi_W(v))$.

Ax. 1: *(Projections.)* For $W \in \mathcal{G}$, π_W must be uniformly coarsely Lipschitz and have uniformly quasi-convex image with uniformly bounded point images.

Ax. 6: *(Large links.)* There exist E, λ such that for $W \in \mathcal{G}$ and $u, v \in X$, there is $\{T_i\} \subset \mathcal{G}^*_W$ of size at most $N = \lambda d_W(u, v) + \lambda$ such that *(i)* $d_W(u, \rho^{T_i}_W) \leq N$ and *(ii)* if $d_W(u, v) \geq E$ for $T \in \mathcal{G}^*_W$, then $T \sqsubseteq T_i$ for some i.

Ax. 9: *(Uniqueness.)* For every $K \geq 0$, there exists $K' \geq 0$ such that if $u, v \in X$ such that $d_W(u, v) \leq K$ for all $W \in \mathcal{G}$, then $d(u, v) \leq K'$.

The *rank* ν of a hierarchically hyperbolic space $(X, \mathcal{G})$ is the largest cardinality of a collection of pairwise orthogonal elements $\{U_j\}$ in $\mathcal{G}$ for which $\pi_{U_j}(X)$ is unbounded; if $(X, \mathcal{G})$ is *asymphoric*, then there exists $C \geq 0$ such that ν is likewise the maximum cardinality of a pairwise orthogonal set for which $\mathrm{diam}\, \mathcal{C}U_j > C$. δ-hyperbolic spaces are hierarchically hyperbolic with respect to a trivial (hence rank at most 1) hierarchical structure; conversely, any rank 1 hierarchically hyperbolic space is δ-hyperbolic if it is also asymphoric [BHS21, Cor. 2.15].

For $\mathcal{A}$ a cocompact multiarc and curve graph and a witness W, the action of $\mathrm{PMod}(\Sigma)$ implies $\pi_W(\mathcal{A})$ is either unbounded or $\mathcal{C}W$ is diameter at most 2, if W is disconnected. Hence Theorem 1.3 implies that cocompact multiarc and curve graphs are asymphoric hyperbolically hierarchical spaces, and the above and that orthogonality is equivalent to disjointness implies that Conjecture 1.6 holds.

Let $d_W(a,b) := \operatorname{diam}_{\mathcal{C}W}(\pi_W(a) \cup \pi_W(b))$. As with $\mathcal{MC}(\Sigma)$, hierarchically hyperbolic spaces have a distance formula [BHS19, Thm. 4.5]:

Theorem 2.4 (Behrstock-Hagen-Sisto)**.** *Let $(X, \mathcal{G})$ be hierarchically hyperbolic. Then there exists C' such that for any $C > C'$, there exist $K, E \geq 0$ such that*

$$d(a,b) \overset{K,E}{=} \sum_{W \in \mathcal{G}} [d_W(a,b)]_C \, .$$

Conjecture 1.7 follows in the cocompact case from Theorems 2.4 and 1.3. //

2.4. **Witnesses and hierarchical hyperbolicity.** While in the literature witnesses are typically assumed to be *connected* essential non-pants subsurfaces (see *e.g.* the "holes" in [MS10] and [Sch]), our setting requires some additional care. Let be $\mathcal{A}$ a multiarc and curve graph and $\mathscr{X}$ its witness set. If $(\mathcal{A}, \mathscr{X})$ is hierarchically hyperbolic with respect to subsurface projection, the following must hold:

(1) For any $T, U \in \mathscr{X}$ with $U \subsetneq T$ there exists $W \subsetneq T$ such that if $V \subseteq T$ and $V \cap U = \varnothing$ then $V \subseteq W$ (for V, W likewise in $\mathscr{X}$; see [BHS19, Def. 1.1(3)]).

In particular (1) may necessitate *disconnected* witnesses, *e.g.* if the components of $T \setminus U$ are in $\mathscr{X}$, but the only essential connected subsurface containing $T \setminus U$ is T. The components of these witnesses should not be pants: since $\mathcal{C}\Sigma_0^3 = \varnothing$, such components cause redundancy in the index set. Technically speaking, we requre:

(2) If $U \pitchfork V$, then the function $x \mapsto \min\{d_U(x, \rho_U^V), d_V(x, \rho_V^U)\}$ for $x \in \mathcal{A}$ is uniformly bounded (see [BHS19, Def. 1.1(4)]).

Hence if $U \neq V \in \mathscr{X}$ and $U \cong (U \cap V) \sqcup \Sigma_0^3 \cong V$, then $\mathcal{C}U = \mathcal{C}V = \mathcal{C}(U \cap V)$ and (2) is satisfied if and only if $\pi_U = \pi_V$ has bounded image.

Definition 2.1 is somewhat stronger than needed: if the components of a witness were required only to be non-pants (and their union, but not necessarily each, intersect every vertex), then Theorem 1.3 would still hold. Nonetheless, our convention gives a more minimal index set satisfying (1) and (2) and provides that the subset of connected witnesses uniquely determines (generates by disjoint unions) a given witness set, which will be convenient in Section 3.1.1.

Remark. While in [BHS19] the index set for $\mathcal{MC}(\Sigma)$ is specified only as essential subsurfaces, we presume that subsurfaces with pants components are excluded.

2.5. **Cocompact actions on graphs.** Let G be a group and let a *G-graph* be a graph equipped with a graph action of G. Likewise, let a *G-space* be a metric space equipped with an isometric G-action. We consider *coarsely G-equivariant maps*, that is maps between G-spaces which are equivariant up to uniform error. These maps will be essential for succintly proving Theorem 1.4.

Much as the category of compact spaces and Lipschitz maps is not particularly complicated, the category of cocompact G-graphs and coarsely equivariant Lipschitz maps is easily understood. In particular, up to bounded difference between maps, it is a pre-order determined only by the existence of a coarsely equivariant map, which is characterized by the following criterion on stabilizer orbits:

Lemma 2.5. *Let G be a group, and let Γ be a connected cocompact G-graph and let Y be a metric G-space. There exists a coarsely equivariant map $\xi : \Gamma \to Y$ if and only if the vertex stabilizers for Γ have bounded orbits in Y, in which case ξ is coarsely Lipschitz and unique up to bounded error.*

Remark. We prove the lemma for $V(\Gamma)$ with the induced metric, noting that $V(\Gamma) \hookrightarrow \Gamma$ is an equivariant quasi-isometry. When Y is geodesic, we can extend any coarsely equivariant Lipschitz map on $V(\Gamma)$ to one on Γ and upgrade ξ to a Lipschitz map.

Proof. Suppose that $\xi : V(\Gamma) \to Y$ is coarsely equivariant. Hence for $v \in V(\Gamma)$ and $g \in \operatorname{stab}_G(v)$, $g\xi(v)$ must be uniformly close to $\xi(gv) = \xi(v)$, hence $\operatorname{stab}_G(v) \cdot \xi(v)$ is bounded. Since G acts isometrically on Y, every $\operatorname{stab}_G(v)$-orbit in Y is bounded. Conversely, fix (finitely many) representatives $v_i \in V(\Gamma)/G$, and for each choose $y_i \in Y$. Let $M = \max_i(\operatorname{diam}(\operatorname{stab}_G(v_i) \cdot y_i))$. For each v_i, choose a transversal $g_{i,k}$ for $\operatorname{stab}_G(v_i)$ over G and let $\gamma_i(g) = g_{i,k}$ for $g^{-1}g_{i,k} \in \operatorname{stab}_G(v_i)$. Let $\xi(v_i) := y_i$ and $\xi(gv_i) := \gamma_i(g)y_i$. For any $v = hv_i \in V(\Gamma)$ note that $(gh)^{-1}g\xi(v) = h^{-1}\xi(hv_i)$ and $(gh)^{-1}\xi(gv) = (gh)^{-1}\xi(ghv_i)$ are in $\operatorname{stab}_G(v_i) \cdot y_i$, hence $d_Y(g\xi(v), \xi(gv)) \leq M$ and ξ is M-coarsely equivariant.

Finally, suppose $\xi : V(\Gamma) \to Y$ is a M-coarsely equivariant map. For $L = \max_{(v,v') \in E(\Gamma)/G}(d_Y(\xi(v), \xi(v'))) + 2M$, ξ is L-Lipschitz, and for any M'-coarsely equivariant map ξ', ξ is C-close for $C = \max_{v \in V(\Gamma)/G}(d_Y(\xi(v), \xi'(v))) + M + M'$. □

Remark 2.6. If Y is cobounded, then ξ is likewise coarsely surjective. In particular, coarse equivariance implies $\operatorname{im} \xi$ is coarsely dense in $G \cdot \operatorname{im} \xi$, which is G-invariant hence coarsely dense in Y.

From Lemma 2.5, we immediately obtain the following important fact:

Corollary 2.7. *Let Γ, Δ be connected cocompact G-graphs. Then Γ, Δ are coarsely equivariantly quasi-isometric if and only if the vertex stabilizers of Γ have bounded orbits in Δ and* vice versa. □

For multiarc and curve graphs, Section 3.1 shows this stabilizer condition is equivalent to the agreement of connected witness sets, whence Theorem 1.4 follows.

3. Hierarchical hyperbolicity of partial marking graphs

We begin with a minor generalization of the complete, clean markings defined in [MM00]. Given an essential simple closed curve $a \subset \Sigma$, let $\mathcal{C}(a) := \mathcal{C}A$ for an annulus A with core a.

Definition 3.1. A *marking* $\mu = \{(a_i, t_i)\}$ on Σ is an essential simple multicurve $\{a_i\}$, denoted $\operatorname{base}(\mu)$, along with a collection of (possibly empty) diameter 1 subsets $t_i \subset \mathcal{C}(a_i)$, denoted $\operatorname{trans}(\mu)$; for $a_i \in \operatorname{base}(\mu)$, let $\operatorname{trans}(\mu, a_i) = t_i$ denote the associated *transversal.*

A *submarking* is a subset $\omega \subset \mu$, regarded as sets of tuples. We say b is a *clean transverse curve* for a curve a if the subsurface F filled by $a \cup b$ has complexity $\xi = 1$ and a, b are adjacent in $\mathcal{C}F$.

Definition 3.2. A marking μ is *locally complete* if whenever $t_i \neq \varnothing$ then the complementary component of $\mathrm{base}(\mu) \setminus \{a_i\}$ containing a_i has complexity $\xi = 1$; if in addition $t_i \neq \varnothing$ implies $t_i = \pi_{a_i}(b_i)$ for some clean transverse curve b_i intersecting $\mathrm{base}(\mu)$ only in a_i, then μ is *(locally) clean.*

Remark 3.3. Given a marking μ, let $\mu' \subset \mu$ denote the submarking with non-empty transversals. Then μ is locally complete (respectively locally clean) if and only if there exists a subsurface $F \supset \mu'$ such that $\partial F \subset \mathrm{base}(\mu) \cup \partial\Sigma$ and μ' is a a complete (respectively clean) marking on F, in the original sense of [MM00]. When we say a marking is *clean*, we will always mean locally clean.

Just as with complete markings, given a locally complete marking μ, a locally clean marking μ' is *compatible* with μ if $\mathrm{base}(\mu) = \mathrm{base}(\mu')$ and, for all $a \in \mathrm{base}(\mu)$, $\mathrm{trans}(\mu', a) = \varnothing$ if and only if $\mathrm{trans}(\mu, a) = \varnothing$ and $d_a(\mathrm{trans}(\mu, a), \mathrm{trans}(\mu', a))$ is minimal among all possible choices of transversal. We have the following from Remark 3.3 and [MM00, Lem. 2.4]:

Lemma 3.4. *For any locally complete marking μ, there exist at least one and at most 4^b compatible clean markings μ', where b is the number of components in μ with non-empty transversal. Furthermore, for any such μ' and $a \in \mathrm{base}(\mu)$, $d_a(\mathrm{trans}(\mu, a), \mathrm{trans}(\mu', a)) \leq 3$.* □

Definition 3.5. Let $\mu = \{(a_j, t_j)\}, \nu = \{(b_k, s_k)\}$ be two markings on Σ. Then their *geometric intersection number* $i(\mu, \nu)$ is defined as follows:

$$
\begin{aligned}
i(\mu,\nu) := {} & i(\mathrm{base}(\mu), \mathrm{base}(\nu)) \\
& + \sum_j \mathrm{diam}_{\mathcal{C}(a_j)} \left[\pi_{a_j}(\mathrm{base}(\nu)) \cup t_j\right] + \sum_k \mathrm{diam}_{\mathcal{C}(b_k)} \left[\pi_{b_k}(\mathrm{base}(\mu)) \cup s_k\right] \\
& + \sum_{a_j = b_k} \mathrm{diam}_{\mathcal{C}(a_j)}(t_j \cup s_k)
\end{aligned}
$$

Definition 3.6. A *partial marking graph* $\mathcal{M}$ on Σ is a simplicial graph whose vertices are clean markings. $\mathcal{M}$ is *cocompact* if it is connected, $\mathrm{PMod}(\Sigma)$ preserves $V(\mathcal{M})$, and this action extends to a cocompact action on $\mathcal{M}$.

As with arc and curve systems, for any $L \geq 0$, any finite-index subgroup of $\mathrm{Mod}(\Sigma)$ acts cofinitely on the set of pairs of clean markings (μ, ν) with $i(\mu, \nu) \leq L$. It follows that the action of $\mathrm{PMod}(\Sigma)$ on $\mathcal{M}$ is cocompact (has finitely many edge orbits) if and only if the edge geometric intersection numbers $\{i(\mu, \nu) : (\mu, \nu) \in E(\mathcal{M})\}$ are bounded. We again obtain an equivalent definition:

Definition 3.7. A connected partial marking graph $\mathcal{M}$ is *cocompact* if $\mathrm{PMod}(\Sigma)$ acts naturally on $\mathcal{M}$ and there exists $L \geq 0$ such that $i(\mu, \nu) \leq L$ for $(\mu, \nu) \in E(\mathcal{M})$.

Note that cocompact multicurve graphs are likewise cocompact partial marking graphs: we need only endow each multicurve with empty transversals. The marking complex $\mathcal{MC}(\Sigma)$ in [MM00] is a cocompact partial marking graph, whose vertices are the complete clean markings on Σ.

Definition 3.8. Let $\mathcal{M}$ be a partial marking graph on Σ. An essential subsurface $W \subset \Sigma$ is a *witness* for $\mathcal{M}$ if for each component $W_0 \subset W$, $W_0 \not\cong \Sigma_0^3$ and for every marking $\mu \in V(\mathcal{M})$ either (i) W_0 intersects base(μ) or (ii) W_0 is an annulus whose core is a component in base(μ) with non-empty transversal.

Remark 3.9. Let $W \subsetneq W'$ be connected essential non-pants subsurfaces. If a clean marking μ satisfies (i) or (ii) for W, then it satisfies (i) for W': in particular, if W is an annulus about $a \in \mathrm{base}(\mu)$ with non-empty transversal and W' is disjoint from base(μ), then W' is necessarily a pair of pants. The connected witnesses of $\mathcal{M}$ are thus closed under *enlargement*: if W is a witness for $\mathcal{M}$, then W' is likewise.

Subsurface projection extends to markings, as defined in [MM00]:

Definition 3.10 (Subsurface projection)**.** Let $\mathcal{M}$ be a partial marking graph on Σ and let $S \subset \Sigma$ be an essential subsurface. If S is connected, define $\pi_S : \mathcal{M} \to 2^{\mathcal{C}S}$ as follows: for $\mu \in V(\mathcal{M})$, if S is an annulus with core $a \in \mathrm{base}(\mu)$ then $\pi_S(\mu) := \mathrm{trans}(\mu, a)$, and otherwise $\pi_S(\mu) := \pi_S(\mathrm{base}(\mu))$ is the usual multicurve subsurface projection. If $S = S' \sqcup S''$, then let $\pi_S(\mu) := \pi_{S'}(\mu) \cup \pi_{S''}(\mu)$.

Remark 3.11. For any essential subsurface $S \subset \Sigma$, $\mathrm{diam}_{\mathcal{C}S}(\pi_S(\mu)) \leq 2$ by *e.g.* [MM00, Lem. 2.3], and more generally $d_S(\mu, \nu) := \mathrm{diam}_{\mathcal{C}S}(\pi_S(\mu) \cup \pi_S(\nu))$ is bounded uniformly in terms of $i(\mu, \nu)$ (see *e.g.* [Wat16, Thm. 2.10]).

Remark 3.12. We note that $\pi_W(\mu) \neq \varnothing$ for all $\mu \in V(\mathcal{M})$ if and only if W is a witness for $\mathcal{M}$. If $\mathcal{M}$ is cocompact then its edge intersection numbers are bounded, hence for any witness W, Remark 3.11 implies π_W is uniformly Lipschitz.

Remark 3.13. Let $S \subset \Sigma$ be an essential subsurface and let $\mathrm{Mod}(S, \partial S)$ act on $\mathcal{M}$ via the homomorphism $\mathrm{Mod}(S, \partial S) \to \mathrm{PMod}(\Sigma)$ obtained by extending by identity. Then π_S is $\mathrm{Mod}(S, \partial S)$-equivariant.

Theorem 3.14. *Let $\mathcal{M}$ be a cocompact partial marking graph, and let $\mathscr{X}$ denote its collection of witnesses. Then $(\mathcal{M}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to subsurface projection to witness curve graphs $\mathcal{C}W$, $W \in \mathscr{X}$.*

We follow the overall strategy in [Vok22]. We first show that the quasi-isometry type of an cocompact partial marking graph $\mathcal{M}$ is fully determined by its set of witnesses $\mathscr{X}$; in particular, there exists a canonical "maximal" representative $\mathcal{M} \xrightarrow{\text{q.i.}} \mathcal{L}_{\mathscr{X}}$, which we then conclude to have the desired hierarchically hyperbolic structure.

3.1. **Equivariant quasi-isometry types.** We establish a bijection between the coarsely $\mathrm{PMod}(\Sigma)$-equivariant quasi-isometry types of cocompact partial marking graphs on Σ and certain collections of connected essential subsurfaces of Σ. Let $[\mathcal{M}]$ denote the coarsely $\mathrm{PMod}(\Sigma)$-equivariant quasi-isometry class of $\mathcal{M}$.

Definition 3.15. A set of connected essential non-pants subsurfaces $\hat{\mathscr{X}}$ is an *admissible connected witness set* for Σ if it is closed under the action of $\mathrm{PMod}(\Sigma)$ and enlargement: if W' is an essential connected non-pants subsurface and $W' \supset W \in \hat{\mathscr{X}}$, then $W' \in \hat{\mathscr{X}}$. An *admissible witness set* $\mathscr{X}$ is a set of subsurfaces generated by disjoint unions over an admissible connected witness set $\hat{\mathscr{X}} \subset \mathscr{X}$.

Given an admissible witness set $\mathscr{X}$, we denote by $\hat{\mathscr{X}} \subset \mathscr{X}$ the subset of connected subsurfaces, which is the unique admissible connected witness set generating $\mathscr{X}$. If $\mathcal{G}$ is a partial marking graph or multiarc and curve graph preserved by $\mathrm{PMod}(\Sigma)$, then its witness subsurfaces comprise an admissible witness set, denoted $\mathscr{X}^{\mathcal{G}}$. More generally, any collection of essential subsurfaces without pants components may be closed to an admissible witness set.

Theorem 3.16. *The map $\mathcal{M} \mapsto \mathscr{X}^{\mathcal{M}}$ induces a bijection $[\mathcal{M}] \mapsto \hat{\mathscr{X}}^{\mathcal{M}}$ between coarsely* $\mathrm{PMod}(\Sigma)$*-equivariant quasi-isometry types of cocompact partial marking graphs and admissible connected witness sets.*

We begin with the following claim, which (among other applications) will show that the putatively bijective map in Theorem 3.16 is well-defined and injective.

Proposition 3.17. *Let each of $\mathcal{M}, \mathcal{M}'$ be a cocompact partial marking graph or multiarc and curve graph on Σ. The vertex stabilizers of $\mathcal{M}$ have bounded orbits in $\mathcal{M}'$ if and only if $\hat{\mathscr{X}}^{\mathcal{M}} \supset \hat{\mathscr{X}}^{\mathcal{M}'}$.*

For convenience, let $\mathrm{base}(\mu) := \mu$ if μ is an arc and curve system. Also note that while Remarks 3.12 and 3.13 were stated for partial marking graphs, analogous arguments hold for subsurface projections of multiarc and curve graphs.

Proof. Note that if $\mu \in V(\mathcal{M}) \cup V(\mathcal{M}')$ and $S \subset \Sigma$ is an essential non-pants subsurface such that $\pi_S(\mu) = \varnothing$, then $\mathrm{Mod}(S, \partial S) \leq \mathrm{stab}(\mu)$. In particular, $\mathrm{Mod}(S, \partial S)$ is supported disjointly from $\mathrm{base}(\mu)$ and, since S is non-pants, disjointly from any clean transverse curve. Hence to show the forward direction, suppose that $W \in \hat{\mathscr{X}}^{\mathcal{M}'} \setminus \hat{\mathscr{X}}^{\mathcal{M}}$ and fix $\mu \in V(\mathcal{M})$ such that $\pi_W(\mu) = \varnothing$. Then $\mathrm{Mod}(W, \partial W) \leq \mathrm{stab}(\mu)$. But $\mathrm{Mod}(W, \partial W)$ has unbounded orbits in $\mathcal{C}W$ and $\pi_W : \mathcal{M}' \to \mathcal{C}W$ is $\mathrm{Mod}(W, \partial W)$-equivariant and Lipschitz, hence $\mathrm{stab}(\mu) \geq \mathrm{Mod}(W, \partial W)$ has unbounded orbits in $\mathcal{M}'$.

Conversely, suppose that $\hat{\mathscr{X}}^{\mathcal{M}} \supset \hat{\mathscr{X}}^{\mathcal{M}'}$ and let $\mu \in V(\mathcal{M})$. Let $K \leq \mathrm{stab}(\mu)$ be a finite-index subgroup that permutes neither the components nor complementary components of $\mathrm{base}(\mu)$. Then $K \cong \prod_i K_i$ for non-trivial subgroups $K_i < \mathrm{PMod}(\Sigma)$ supported on disjoint subsurfaces $S_i \subset \Sigma \setminus \mathrm{base}(\mu)$, such that each S_i is either a non-annular, non-pants component or an annulus peripheral in a complementary pair of pants. For each S_i, we claim that $\pi_{S_i}(\mu) = \varnothing$, hence $S_i \notin \hat{\mathscr{X}}^{\mathcal{M}}$. In particular, since S_i is disjoint from $\mathrm{base}(\mu)$, if $\pi_{S_i}(\mu) \neq \varnothing$ then S_i must be an annulus parallel to a component of μ with non-empty transversal; however, in this case $K_i \leq \mathrm{stab}(\mu)$ does not fix μ. It follows that each S_i is not a witness for $\mathcal{M}'$, hence K_i has a fixed point in $\mathcal{M}'$. We obtain that K (and thus $\mathrm{stab}(\mu)$) has bounded orbits in $\mathcal{M}'$. $\square$

In particular, by Corollary 2.7 $\mathcal{M}$ and $\mathcal{M}'$ are coarsely $\mathrm{PMod}(\Sigma)$-equivariantly quasi-isometric if and only if $\hat{\mathscr{X}}^{\mathcal{M}} = \hat{\mathscr{X}}^{\mathcal{M}'}$. Theorem 1.4 follows in the case that $\mathcal{M}, \mathcal{M}'$ are both multiarc and curve graphs. //

Remark. Since witness sets are generated by their connected witness subsets, $\hat{\mathscr{X}}^{\mathcal{M}} \supset \hat{\mathscr{X}}^{\mathcal{M}'}$ if and only if $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$, hence either condition is equivalent in the above.

Remark 3.18. By Lemma 2.5 $\hat{\mathscr{X}}^{\mathcal{M}} \supset \hat{\mathscr{X}}^{\mathcal{M}'}$ (equivalently $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}^{\mathcal{M}'}$) if and only if there is a coarsely equivariant map $\xi : \mathcal{M} \to \mathcal{M}'$; ξ is then coarsely unique and Lipschitz, and by Remark 2.6, coarsely surjective.

3.1.1. *A universal partial marking graph.* It remains to show that the induced map in Theorem 3.16 is surjective. Following [Vok22], for any admissible witness set $\mathscr{X}$ we construct a cocompact partial marking graph $\mathcal{L}_{\mathscr{X}}$ such that $\hat{\mathscr{X}}^{\mathcal{L}_{\mathscr{X}}} = \hat{\mathscr{X}}$. $\mathcal{L}_{\mathscr{X}}$ will furnish us with both an explicit representative in each coarsely equivariant quasi-isometry class, as well as a canonical choice of coarsely equivariant Lipschitz map $\mathcal{M} \to \mathcal{L}_{\mathscr{X}}$, whenever one exists. Both properties will be useful for proving Theorem 3.14.

We first extend the notion of an *elementary move* on a clean marking, as introduced in [MM00]:

Definition 3.19. Let μ be a clean marking on Σ, and suppose a clean marking μ' is obtained from μ by either

(i) *a twist:* replace some component $(a, \pi_a(b)) \in \mu$ with $(a, \pi_a(\tau_a b))$, for τ_a a Dehn twist or a half Dehn twist about a.
(ii) *a flip:* replace some component $(a, \pi_a(b)) \in \mu$ with $(b, \pi_b(a))$ and choose a clean marking compatible with the result.

Then μ' is obtained from μ by an *elementary move*. (We assume $i(a, b) > 0$.)

Remark. While not canonical in a strict sense, by Lemma 3.4 flip moves are unique up to finitely many choices for fixed a, b, all uniformly close after projection to witnesses.

Definition 3.20. Let $\mathscr{X}$ be an admissible witness set on Σ. Define $\mathcal{L}_{\mathscr{X}}$ to be the simplicial graph whose vertices are all clean markings that intersect every surface in $\mathscr{X}$, in the sense of Definition 3.8, and for which $(\mu, \nu) \in E(\mathcal{L}_{\mathscr{X}})$ if and only if μ is obtained from ν by either

(i) adding or removing a component (a, t), or
(ii) adding or removing a transversal, or
(iii) an elementary move.

Lemma 3.21. *Let $\mathscr{X}$ be an admissible witness set. $\mathscr{X}^{\mathcal{L}_{\mathscr{X}}} = \mathscr{X}$ and $\hat{\mathscr{X}}^{\mathcal{L}_{\mathscr{X}}} = \hat{\mathscr{X}}$.*

Proof. Admissible witness sets are generated by disjoint unions over their connected subsets, hence it suffices to show $\hat{\mathscr{X}}^{\mathcal{L}_{\mathscr{X}}} = \hat{\mathscr{X}}$. $\hat{\mathscr{X}} \subset \hat{\mathscr{X}}^{\mathcal{L}_{\mathscr{X}}}$ since by definition the vertices of $\mathcal{L}_{\mathscr{X}}$ intersect every subsurface in $\mathscr{X} \supset \hat{\mathscr{X}}$. Conversely, suppose $W \notin \hat{\mathscr{X}}$ is a connected essential subsurface of Σ without pants components. $\hat{\mathscr{X}}$ is closed under enlargement, hence W contains no subsurfaces in $\hat{\mathscr{X}}$. Let μ be the clean marking on Σ obtained from any complete, clean marking on $\Sigma \setminus W$ by adding distinct components in ∂W with empty transversals. Then $\mu \in V(\mathcal{L}_{\mathscr{X}})$ and does not intersect W. $W \notin \hat{\mathscr{X}}^{\mathcal{L}_{\mathscr{X}}}$. $\square$

It is clear that $\mathrm{PMod}(\Sigma)$ acts naturally on $\mathcal{L}_{\mathscr{X}}$ and that two markings are adjacent only if their geometric intersection is less than some uniform constant L.

We first show that $\mathcal{L}_{\mathscr{X}}$ is connected, hence cocompact. In addition, $\mathcal{L}_{\mathscr{X}}$ is universal in the following sense: suppose that $\mathcal{M}$ is a cocompact marking graph such that $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}$. Then every marking in $\mathcal{M}$ intersects every witness in $\mathscr{X}$, hence $V(\mathcal{M}) \subset V(\mathcal{L}_{\mathscr{X}})$ and is preserved by the action $\mathrm{PMod}(\Sigma)$. Given the connectedness of $\mathcal{L}_{\mathscr{X}}$, extend this inclusion to a coarsely $\mathrm{PMod}(\Sigma)$-equivariant map $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ by mapping edges to geodesic paths.

We note that $\mathcal{MC}(\Sigma)$ admits every essential subsurface without pants components as a witness, hence $\mathscr{X}(\mathcal{MC}(\Sigma)) \supset \mathscr{X}$. Since the edges of $\mathcal{MC}(\Sigma)$ correspond to elementary moves, ι is in fact a (simplicial) embedding and $\mathcal{MC}(\Sigma)$ is a connected $\mathrm{PMod}(\Sigma)$-invariant full subgraph of $\mathcal{L}_{\mathscr{X}}$. Hence to show that $\mathcal{L}_{\mathscr{X}}$ is connected, it suffices to prove that every marking lies in the same component as $\mathcal{MC}(\Sigma)$, as the following lemma shows:

Lemma 3.22. *Any marking μ in $V(\mathcal{L}_{\mathscr{X}})$ may be completed to a complete, clean marking μ' through a sequence of length at most $2\xi(\Sigma) - |\mu|$ of adding components and transversals.*

Proof. We note that we may always add components with empty transversal: since μ is already locally complete, no new base curve intersects existing clean transverse curves. Likewise, we may always add (clean) transversals to complete markings. Add $\xi(\Sigma) - |\mu|$ components with empty transversal to obtain the complete marking μ''. For at most $\xi(\Sigma)$ components with empty transversal in μ'', add transversals to obtain a complete, clean marking $\mu' \in \mathcal{MC}(\Sigma)$. □

Since $\mathcal{L}_{\mathscr{X}}$ is cocompact and (for $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}$) the map $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ is coarsely $\mathrm{PMod}(\Sigma)$-equivariant, ι is coarsely identical to ξ in Remark 3.18. We obtain:

Lemma 3.23. *Let $\mathscr{X}$ be an admissible witness set and $\mathcal{M}$ a cocompact marking graph such that $\mathscr{X}^{\mathcal{M}} \supset \mathscr{X}$. The universal map $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ is Lipschitz, coarsely surjective, and coarsely canonical. When $\mathscr{X}^{\mathcal{M}} = \mathscr{X}$, it is a quasi-isometry.* □

Moreover, since ι is an inclusion on vertices, $\pi_W = \pi_W \circ \iota$ for any $W \in \mathscr{X}$. Remark 3.18 then implies:

Proposition 3.24. *Any coarsely* $\mathrm{PMod}(\Sigma)$*-equivariant map between cocompact partial marking graphs coarsely preserves witness subsurface projections.*

3.2. **Hierarchical structure of $(\mathcal{L}_{\mathscr{X}}, \mathscr{X})$.** We endow $\mathcal{L}_{\mathscr{X}}$ with the usual hierarchical structure via witness subsurface projections $\{\pi_W : \mathcal{L}_{\mathscr{X}} \to 2^{\mathcal{C}W}\}_{W \in \mathscr{X}}$ and the following relations on $\mathscr{X}$:

- $U \sqsubseteq V$ if and only if $U \subset V$ up to isotopy; and
- $U \perp V$ if and only if U, V are disjoint up to isotopy.

Let $U \pitchfork V$ if neither of the above hold. For $W \in \mathscr{X}$, π_W is uniformly quasiconvex: any curve (potentially adding transversal) may be completed to a marking in $V(\mathcal{L}_{\mathscr{X}})$, thus π_W is surjective if W is not an annulus; if W is an annulus with core curve a, then any orbit of a Dehn twist about a projects to a 1-coarsely dense subset of $\mathcal{C}W$. Hence π_W satisfies Axiom 1 by the above and Remark 3.12.

If $U \sqsubseteq V$, then let $\rho^V_U = \pi_U : \mathcal{C}V \to 2^{\mathcal{C}U}$, and if in addition $U \neq V$, let $\rho^U_V = \partial_V U$, the non-peripheral boundary of U in V. If $U \pitchfork V$, let $\rho^V_U = \pi_U(\partial V)$. Axioms 2 and 3 follow immediately from the definition of witness subsurfaces and our choice of relations and associated projections, and Axioms 4,5,7 and 8 follow from the fact that the same hold for $\mathcal{MC}(\Sigma)$, which lies as a $2\xi(\Sigma)$-coarsely dense subcomplex of $\mathcal{L}_{\mathscr{X}}$, and that the π_W are uniformly Lipschitz. Axiom 6 follows similarly, along with the fact that witness sets are closed under enlargement:

Lemma 3.25. *For $W \in \mathscr{X}$ and $\mu, \nu \in V(\mathcal{MC}(\Sigma)) \subset V(\mathcal{L}_{\mathscr{X}})$, if $\{T_i\} \subset \mathscr{X}^{\mathcal{MC}(\Sigma)}$ satisfies Axiom 6 for $\mathcal{MC}(\Sigma)$, then $\{T_i\} \cap \mathscr{X}$ satisfies Axiom 6 for $\mathcal{L}_{\mathscr{X}}$ with the same constants.*

In particular, if $\pi_T(\mu, \nu)$ is large for $T \in \mathscr{X}^*_W$, then T is contained in some T_i, which implies $T_i \in \mathscr{X}^*_W$. As above, we obtain that Axiom 6 holds for any $\mu, \nu \in \mathcal{L}_{\mathscr{X}}$ up to uniformly modifying constants. We verify Axiom 9 below.

Definition 3.26. Given an admissible witness set $\mathscr{X}$, let the *twist-free witness set* $\tilde{\mathscr{X}} \subset \mathscr{X}$ denote the admissible witness set with all annular witnesses removed.

Definition 3.27. If $\mathscr{X}$ is an admissible witness set without annuli, let $\mathcal{K}_{\mathscr{X}}$ denote the full subgraph of $\mathcal{L}_{\mathscr{X}}$ spanned by markings with empty transversals, with additional edges (μ, ν) corresponding to *twist-free flip moves*: ν is obtained from μ by replacing a component $(a, \varnothing) \mapsto (b, \varnothing)$ such that, if F is the subsurface filled by a, b, then F is connected, $\partial F \subset \mathrm{base}(\mu) \cup \partial\Sigma$, and $d_{\mathcal{C}F}(a, b) = 1$.

In the following, a *witness curve* is one parallel to an annular witness. Axiom 9, in our setting, reads as follows:

Lemma 3.28. *For any $K \geq 0$, there exists $K' \geq 0$ such that for any $\mu, \nu \in V(\mathcal{L}_{\mathscr{X}})$, if $d_W(\mu, \nu) \leq K$ for all $W \in \mathscr{X}$, then $d_{\mathcal{L}_{\mathscr{X}}}(\mu, \nu) \leq K'$.*

Proof. We regard $\mathcal{K}_{\tilde{\mathscr{X}}}$ as a (twist-free) multicurve graph. For any multicurve $m \in V(\mathcal{K}_{\tilde{\mathscr{X}}})$, there exists a clean marking $\mu \in V(\mathcal{L}_{\mathscr{X}})$ such that $\mathrm{base}(\mu) = m$. In particular every non-annular witness in $\mathscr{X}$ intersects m, and if A is some annular witness disjoint from m with core curve a, then any component of $\Sigma \setminus m$ containing a must be a pair of pants, else a non-annular witness. Hence $a \in m$ and any complementary component for m incident to a is a pair of pants: we may choose a clean transverse curve b for a disjoint from $m \setminus \{a\}$. Let μ consist of components $(a, \pi_a(b))$ for $a \in m$ if a is parallel to an annular witness, and $(a, \varnothing)$ otherwise.

By [Vok22, Prop. 3.6], there exists a path of multicurves $\tilde{P} \subset \mathcal{K}_{\tilde{\mathscr{X}}}$ between $\mathrm{base}(\mu), \mathrm{base}(\nu)$ of length ℓ at most $K'' = K''(K, \mathscr{X})$. Let $\tilde{P} = (m_j)$ with $m_1 = \mathrm{base}(\mu)$ and $m_\ell = \mathrm{base}(\nu)$. Let $P_0 = (\omega_j) \in V(\mathcal{L}_{\mathscr{X}})$ be a sequence of clean markings such that $\omega_1 = \mu, \omega_\ell = \nu$, and $\mathrm{base}(\omega_j) = m_j$, where $\omega_{j \neq 1, \ell}$ is chosen as described above. If $d_{\mathcal{L}_{\mathscr{X}}}(\omega_j, \omega_{j+1})$ is uniformly bounded indepedent of the path $\tilde{P}$, then we conclude by the triangle inequality. In fact, it suffices to control projections to annular witnesses:

Claim 3.29. *Let $\omega, \omega' \in V(\mathcal{L}_{\mathscr{X}})$ such that $(\mathrm{base}(\omega), \mathrm{base}(\omega')) \in E(\mathcal{K}_{\tilde{\mathscr{X}}})$, and let $D \geq 0$ such that $d_c(\omega, \omega') \leq D$ for any $c \in \mathrm{base}(\omega) \cup \mathrm{base}(\omega')$ parallel to an annular witness. Then $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega') \leq (D + 3)(\xi(\Sigma) + 1)$.*

Proof of claim. Without loss of generality assume $|\omega| \leq |\omega'|$, hence base(ω') is obtained from base(ω) by either adding a component b or a twist-free flip $a \mapsto b$. Let ω'' be obtained from ω by adding $(b, \mathrm{trans}(\omega', b))$ or replacing $(a, \mathrm{trans}(\omega, a))$ with $(b, \pi_b(a))$ and choosing a compatible clean marking, as appropriate. In the first case, $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') = 1$, and if a is not parallel to an annular witness, then in the second case $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') \leq 3$: remove and replace the transversal for a with $\pi_a(b)$, then flip. Otherwise, a is a witness curve and at most $d_a(\omega, \omega') \leq D$ twist moves along a suffice to replace $\mathrm{trans}(\omega, a)$ with $\pi_a(b)$, hence $d_{\mathcal{L}_{\mathscr{X}}}(\omega, \omega'') \leq D + 1$.

We show that $d_{\mathcal{L}_{\mathscr{X}}}(\omega'', \omega') \leq (D+3)|\omega''|$ and conclude. base$(\omega'') =$ base(ω'), hence to obtain ω' we need only modify transversals in ω'' to agree with those in ω'. Let $c \in$ base(ω''). As above, at most 2 moves suffice if c is not a witness curve and $d_c(\omega'', \omega')$ otherwise; by Lemma 3.4, $d_c(\omega'', \omega') \leq d_c(\omega, \omega') + 3 \leq D + 3$. □

Let $P = (\omega_j)$ be a sequence of markings in $\mathcal{L}_{\mathscr{X}}$ of length $\ell \leq K''$ such that $\omega_1 = \mu$, $\omega_\ell = \nu$, (base(ω_j)) is a path in $\mathcal{K}_{\tilde{\mathscr{X}}}$, and P is minimal in the following sense: P has the fewest number of pairs (j, a) for which $1 \leq j < \ell$, $a \in$ base$(\omega_j) \cup$ base(ω_{j+1}) is parallel to an annular witness, and $d_a(\omega_j, \omega_{j+1}) > K + c_0 K'' + 6$, where c_0 is a universal constant defined below. P exists by the existence of P_0. We show that if P has such a pair (j, a) then there exists a more minimal sequence P', and conclude by contradiction and the above. Fix a in such a pair and let τ_a denote the Dehn twist about a. With the action of $\mathbb{Z} \cong \langle \tau_a \rangle$ on $\mathcal{C}(a)$, $\mathcal{C}(a)$ is 2-coarsely equivariantly $(1,1)$-quasi-isometric to $\mathbb{Z}$ (see [MM00, §2.4]). Let $m_j =$ base(ω_j). For each $j < \ell$ for which $a \in m_j \cup m_{j+1}$, choose $\sigma_j \in \mathbb{Z}$ such that $d_a(\omega_j, \tau_a^{\sigma_j} \omega_{j+1}) \leq 3$; let $J < \ell$ be the last such index. Set $\sigma_j = 0$ otherwise. Since $d_a(\mu, \nu) = d_a(\omega_1, \omega_\ell) \leq K$, there exists a universal constant $c_0 > 0$ such that

$$\left| \sum_{j=1}^{J} \sigma_j \right| \leq K + c_0(\ell - 1) + 1 \leq K + c_0 K'' \,. \tag{1}$$

Let $\rho_j = \sum_{k=1}^{j-1} \sigma_k$ and let $P' = (\omega'_j)$, where $\omega'_j = \tau_a^{\rho_j} \omega_j$ for $j \leq J$, and $\omega'_j = \omega_j$ otherwise; let $m'_j =$ base(ω'_j). We first verify that (m'_j) is a path in $\mathcal{K}_{\tilde{\mathscr{X}}}$. Since $\mathcal{K}_{\tilde{\mathscr{X}}}$ is preserved by τ_a, for $j < J$ it suffices that $(m_j, \tau_a^{\sigma_j} m_{j+1})$ is an edge in $\mathcal{K}_{\tilde{\mathscr{X}}}$. For $\sigma_j = 0$, this is immediate. Else $a \in m_j \cup m_{j+1}$. If $a \in m_{j+1}$ then $\tau_a^{\sigma_j} m_{j+1} = m_{j+1}$, and otherwise $a \in m_j$ and m_{j+1} is obtained by a twist-free flip $a \mapsto b$: we observe that $\tau_a^{\sigma_j} b$ intersects a minimally in the surface filled by a, b. For $j = J$, it suffices that $(m_J, \tau_a^{-\rho_J} m_{J+1})$ is an edge in $\mathcal{K}_{\tilde{\mathscr{X}}}$: since $a \in m_J \cup m_{J+1}$ by assumption, an identical argument applies.

We show that P' has strictly fewer pairs (j, a') with $a' \in m'_j \cup m'_{j+1}$ a witness curve such that $d_{a'}(\omega'_j, \omega'_{j+1}) > K + c_0 K'' + 6$.

Claim 3.30. *Let ω, ω' be clean markings whose respective base multicurves m, m' differ by adding a component or a (twist-free) flip. Let $a \in m \cup m'$ and $a'' \in m$ be distinct. Then $d_{a''}(\omega, \tau_a^q \omega') = d_{a''}(\omega, \omega')$ for $q \in \mathbb{N}$.*

Proof of claim. If $a'' \in m \cap m'$, then $\pi_{a''}(\omega) = \mathrm{trans}(\omega, a'')$ and $\pi_{a''}(\omega') = \mathrm{trans}(\omega', a'')$, projections of clean transverse curves b, c respectively. Since $a \in m \cup m'$, $a \cap a'' = \varnothing$ and at most one of b, c intersects a, hence τ_a lifts to an action on $\mathcal{C}(a'')$ that fixes either $\pi_{a''}(\omega)$ or $\pi_{a''}(\omega')$: we obtain $d_{a''}(\omega, \tau_a^q\omega') = d_{a''}(\tau_a^q\omega, \tau_a^q\omega') = d_{a''}(\omega, \omega')$. If instead $a'' \in m \setminus m'$, then $a \in m'$, since otherwise two distinct curves lie in $m \setminus m'$ and m, m' do not differ by adding a component or a flip. Then $\pi_{a''}(\omega') = \pi_{a''}(m')$ is preserved by τ_a and likewise $d_{a''}(\omega, \tau_a^q\omega') = d_{a''}(\omega, \omega')$. □

Observe that the maps $a' \in m'_j \mapsto \tau_a^{-\rho_j}a'$ and $a' \in m'_{j+1} \mapsto \tau_a^{-\rho_{j+1}}a'$ induce a bijection from $m'_j \cup m'_{j+1}$ to $m_j \cup m_{j+1}$ preserving witness curves. We first assume that $j < J$, and suppose that $a' \in m'_j$ is a witness curve. Let $a'' = \tau_a^{-\rho_j}a' \in m_j$. As above, by translating by $\tau_a^{-\rho_j}$ we observe that $d_{a'}(\omega'_j, \omega'_{j+1}) = d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1})$. If $\sigma_j = 0$ then $d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) = d_{a''}(\omega_j, \omega_{j+1})$. Suppose $\sigma_j \neq 0$ and thus $a \in m_j \cup m_{j+1}$. For $a \neq a''$, Claim 3.30 implies that likewise $d_{a''}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) = d_{a''}(\omega_j, \omega_{j+1})$. An analogous argument applies if $a' \in m'_{j+1}$, replacing a'' with $\tau_a^{-\rho_{j+1}}a' \in m_{j+1}$ and translating by $\tau_a^{-\rho_{j+1}}$. Finally if $a'' = a$ then $a' = a$, thus by our choice of σ_j $d_{a'}(\omega_j, \tau_a^{\sigma_j}\omega_{j+1}) \leq 3 \leq K + c_0K'' + 6$.

Let $j = J$. Then $a \in m_J \cup m_{J+1}$ by our choice of J. We apply the arguments above: if $a' \in m'_J$, then let $a'' = \tau_a^{-\rho_J}a' \in m_J$ and translate by $\tau_a^{-\rho_J}$, else if $a' \in m'_{J+1} = m_{J+1}$, then let $a'' = a'$. Hence if $a'' \neq a$, $d_{a'}(\omega'_J, \omega'_{J+1}) = d_{a''}(\omega_J, \omega_{J+1})$. If $a'' = a$ then $a' = a$; observe that by (1), $|\rho_{J+1}| \leq K + c_0K''$, hence $d_a(\omega_{J+1}, \tau_a^{\rho_{J+1}}\omega_{J+1}) \leq K + c_0K'' + 3$. But $d_a(\omega'_J, \tau_a^{\rho_{J+1}}\omega_{J+1}) = d_a(\omega_J, \tau_a^{\sigma_J}\omega_{J+1}) \leq 3$, hence $d_{a'}(\omega'_J, \omega'_{J+1}) \leq K + c_0K'' + 6$. □

Hence $(\mathcal{L}_{\mathscr{X}}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to witness subsurface projections. However, if $(X, \mathcal{G})$ is a hierarchically hyperbolic space and $\psi : X' \to X$ is a quasi-isometry, then $(X', \mathcal{G})$ is likewise a hierarchically hyperbolic space by precomposing projections with ψ (this fact is remarked in *e.g.* [BHS17, §1.1.4]). Thus by Section 3.1.1, Theorem 3.14 follows: for any $\mathcal{M}$ for which $\mathscr{X}^{\mathcal{M}} = \mathscr{X}$, the universal quasi-isometry $\iota : \mathcal{M} \to \mathcal{L}_{\mathscr{X}}$ preserves subsurface projection, hence $(\mathcal{M}, \mathscr{X})$ is a hierarchically hyperbolic space with respect to the projections $\pi_W \circ \iota = \pi_W$, $W \in \mathscr{X}$. //

4. Hierarchical hyperbolicity of multiarc and curve graphs

For a cocompact multiarc and curve graph $\mathcal{A}$ on Σ, we construct a cocompact partial marking graph $\mathcal{M}_{\mathcal{A}}$ on Σ with an identical witness set and a PMod(Σ)-equivariant map $\zeta : \mathcal{A} \to \mathcal{M}_{\mathcal{A}}$ that coarsely preserves witness subsurface projection. By Remark 3.18, ζ is a quasi-isometry and Theorem 1.3 follows from Theorem 3.14.

We note Remark 3.18 also implies $\mathcal{A}$ is quasi-isometric to the cocompact partial marking graph $\mathcal{L}_{\mathscr{X}^{\mathcal{A}}}$. However, *a priori* the induced hierarchically hyperbolic structure may differ from that in Theorem 1.3: without the universal map ι, we cannot be certain the quasi-isometry preserves witness subsurface projection. The existence of ζ confirms the latter, and in particular shows Proposition 3.24 extends to cocompact multiarc and curve graphs.

4.1. **Constructing the associated marking graph.** Given an arc and curve system α, we construct a corresponding clean marking μ as follows (Figure 1). Fix an enumeration $\alpha = \{a_j\}_{j=1}^N$ and let $\alpha_k = \{a_j\}_{j=1}^k$. Let F_k denote the essential subsurface filled by α_k and let $\partial_\Sigma(\alpha_k)$ denote non-peripheral components of ∂F_k; let $F = F_N$. Let $m = \bigcup_k \partial_\Sigma(\alpha_k)$ and obtain μ by adding to each component $c \in m$ the transversal $\pi_c(\alpha)$. F_k is obtained from F_{k-1} by adding an annulus or a pair of pants (if a_k is a curve or arc, respectively), hence $m \setminus \partial F$ is a pants decomposition on F. Since α fills F, it intersects exactly the components of m interior to F and $\mu \setminus \partial F$ is complete on F; by construction $\partial F = \partial F_N \subset m \cup \partial\Sigma$ up to isotopy. It follows that μ is locally complete. Let μ_α be the (finite) collection of all clean markings compatible with μ, for every choice of enumeration of the a_j.

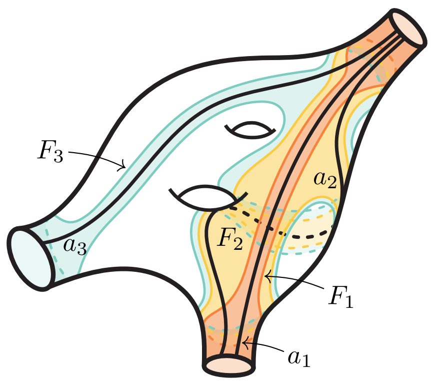

FIGURE 1. Constructing μ from neighborhoods F_k of $\alpha_k = \{a_j\}_{j=1}^k$.

We define the vertices of $\mathcal{M}_\mathcal{A}$ to be

$$V(\mathcal{M}_\mathcal{A}) = \bigcup_{\alpha \in V(\mathcal{A})} \mu_\alpha \,.$$

Let $(\mu, \nu) \in E(\mathcal{M}_\mathcal{A})$ if and only if $\mu \in \mu_\alpha$ and $\nu \in \mu_\beta$ for $(\alpha, \beta) \in E(\mathcal{A})$. Define $\zeta : V(\mathcal{A}) \to V(\mathcal{M}_\mathcal{A})$ to be the coarse map $\alpha \mapsto \mu_\alpha$. In particular, observe $\operatorname{diam}(\zeta(\alpha)) \leq 2$ since $\mathcal{A}$ is connected and not a singleton: for β adjacent to α in $\mathcal{A}$, any $\mu, \mu' \in \zeta(\alpha)$ are adjacent to μ_β in $\mathcal{M}_\mathcal{A}$. Since μ_α is canonical, ζ is $\mathrm{PMod}(\Sigma)$-equivariant; $\mathcal{M}_\mathcal{A}$ is likewise preserved under the action of $\mathrm{PMod}(\Sigma)$ and connected. Let $W \subset \Sigma$ be an essential, non-pants subsurface. For α, μ, and F as above, α is filling and $\mu \setminus \partial F$ is complete on F, and $\mu \cap \partial F$ has empty transversals. Hence W intersects μ if and only if it intersects F if and only if it intersects α: the witness set for $\mathcal{M}_\mathcal{A}$ is identical to that of $\mathcal{A}$. Finally, for α and $\mu \in \mu_\alpha$, α, μ have identical projection to annular witnesses parallel with $\operatorname{base}(\mu)$ by construction. Fixing $\alpha_k \subset \alpha$ filling F_k as above, any other witness W intersects $\partial F_k \setminus \partial\Sigma \subset \operatorname{base}(\mu)$ for some k, hence likewise α_k: since $\alpha_k \cap \partial F_k = \varnothing$, $\alpha, \operatorname{base}(\mu)$ (thus equivalently α, μ) have coarsely equal projection to $\mathcal{C}W$.

4.2. **Counting intersections.** It remains to show that if μ, μ' are adjacent in $\mathcal{M}_\mathcal{A}$ then $i(\mu, \mu')$ is uniformly bounded, hence $\mathcal{M}_\mathcal{A}$ is cocompact. If μ, μ' are adjacent then there exists α, α' adjacent in $\mathcal{A}$ such that $\mu \in \mu_\alpha$ and $\mu' \in \mu_{\alpha'}$. We first show:

Claim 4.1. *$i(\operatorname{base}(\mu), \operatorname{base}(\mu'))$ is uniformly bounded in terms of $i(\alpha, \alpha')$.*

Proof of claim. Fix representatives for α, α' in minimal position, and isotope μ, μ' near $\alpha \cup \partial\Sigma, \alpha' \cup \partial\Sigma$ respectively so that $\mathrm{base}(\mu), \mathrm{base}(\mu')$ intersect only in small pairwise disjoint disk and half disk neighborhoods D_p centered at points p in $\alpha \cap \alpha'$, $\alpha \cap \partial\Sigma$, and $\alpha' \cap \partial\Sigma$. In each neighborhood D_p, we may assume each subarc in $\mathrm{base}(\mu) \cap D$ intersects each subarc in $\mathrm{base}(\mu') \cap D$ at most once, *e.g.* by replacing them with geodesic segments between their endpoints. For $p \in \alpha \cap \alpha'$, each component of $\mathrm{base}(\mu), \mathrm{base}(\mu')$ may contribute two arcs in D_p; for $p \in \alpha \cap \partial\Sigma$, components of $\mathrm{base}(\mu)$ may contribute two arcs and of $\mathrm{base}\, \mu'$ one, and *vice versa* for $p \in \alpha' \cap \partial\Sigma$. Counting intersections near each p, we obtain $i(\mathrm{base}(\mu), \mathrm{base}(\mu')) \leq 4|\mu||\mu'|i(\alpha, \alpha') + 4|\mu||\mu'|(|\alpha| + |\alpha'|) \leq 4\xi(\Sigma)^2(i(\alpha, \alpha') + 2\dim \mathcal{AC}(\Sigma))$. ☐

For $c \in \mathrm{base}(\mu)$, $d_c(\mu, \mu') = d_c(\alpha, \alpha')$ is uniformly bounded in terms of $i(\alpha, \alpha')$, and likewise for $c' \in \mathrm{base}(\mu')$ (*c.f.* Remark 3.11, which applies to arc and curve systems after replacing arc components with nearby disjoint curves). Since $i(\alpha, \alpha')$ is uniformly bounded, so is $i(\mu, \mu')$. //

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References

[BHS17] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, *Hierarchically hyperbolic spaces I: Curve complexes for cubical groups*, Geometry & Topology **21** (2017), no. 3, 1731–1804.

[BHS19] ______, *Hierarchically hyperbolic spaces II: Combination theorems and the distance formula*, Pacific Journal of Mathematics **299** (2019), no. 2, 257–338.

[BHS21] ______, *Quasiflats in hierarchically hyperbolic spaces*, Duke Mathematical Journal **170** (2021), no. 5, 909 – 996.

[BNV22] Assaf Bar-Natan and Yvon Verberne, *The grand arc graph*, 2022.

[CR24] Aaron Calderon and Jacob Russell, *Hierarchical hyperbolicity of admissible curve graphs and the boundary of marked strata*, arXiv preprint, `https://arxiv.org/abs/2409.06798`, 2024.

[Kop23] Michael C. Kopreski, *Prescribed arc graphs*, 2023.

[Kop24] ______, *The asymptotic dimension of the grand arc graph is infinite*, arXiv preprint, `https://arxiv.org/abs/2402.03603`, 2024.

[MM00] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves II: Hierarchical structure*, Geometric & Functional Analysis **10** (2000), 902–974.

[MR20] Kathryn Mann and Kasra Rafi, *Large scale geometry of big mapping class groups*, 2020.

[MS10] Howard A. Masur and Saul Schleimer, *The geometry of the disk complex*, Journal of the American Mathematical Society **26** (2010), 1–62.

[Sch] Saul Schleimer, *Notes on the complex of curves*, `http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf`.

[Sis17] Alessandro Sisto, *What is a hierarchically hyperbolic space?*, 2017.

[Vok22] Kate M. Vokes, *Hierarchical hyperbolicity of graphs of multicurves*, Algebraic & Geometric Topology **22** (2022), no. 1, 113–151.

[Wat16] Yohsuke Watanabe, *Intersection numbers in the curve graph with a uniform constant*, Topology and its Applications **204** (2016), 157–167.